

# CONSTRUCTION OF MULTI-SOLITONS FOR THE ENERGY-CRITICAL WAVE EQUATION IN DIMENSION 5

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**ABSTRACT.** We construct 2-solitons of the focusing energy-critical nonlinear wave equation in space dimension 5, i.e. solutions  $u$  of the equation such that

$$u(t) - [W_1(t) + W_2(t)] \rightarrow 0 \quad \text{as } t \rightarrow +\infty$$

in the energy space, where  $W_1$  and  $W_2$  are Lorentz transforms of the explicit standing soliton  $W(x) = (1 + |x|^2/15)^{-3/2}$ , with any speeds  $\ell_1 \neq \ell_2$  ( $|\ell_k| < 1$ ). The existence result also holds for the case of  $K$ -solitons, for any  $K \geq 3$ , assuming that the speeds  $\ell_k$  are collinear.

The main difficulty of the construction is the strong interaction between the solitons due to the slow algebraic decay of  $W(x)$  as  $|x| \rightarrow +\infty$ . This is in contrast with previous constructions of multi-solitons for other nonlinear dispersive equations (like generalized KdV and nonlinear Schrödinger equations in energy subcritical cases), where the interactions are exponentially small in time due to the exponential decay of the solitons.

## 1. INTRODUCTION

**1.1. Statement of the main result.** We consider the focusing energy-critical nonlinear wave equation in dimension 5

$$\begin{cases} \partial_t^2 u - \Delta u - |u|^{\frac{4}{3}} u = 0, & (t, x) \in [0, \infty) \times \mathbb{R}^5, \\ u|_{t=0} = u_0 \in \dot{H}^1, \quad \partial_t u|_{t=0} = u_1 \in L^2. \end{cases} \quad (1.1)$$

Recall that the Cauchy problem for equation (1.1) is locally well-posed in the energy space  $\dot{H}^1 \times L^2$ , using suitable Strichartz estimates. See e.g. [26, 11, 16, 29, 30, 28, 12, 14]. Note that equation (1.1) is invariant by the  $\dot{H}^1$  scaling: if  $u(t, x)$  is solution of (1.1), then

$$u_\lambda(t, x) = \frac{1}{\lambda^{3/2}} u\left(\frac{t}{\lambda}, \frac{x}{\lambda}\right)$$

is also solution of (1.1) and  $\|u_\lambda\|_{\dot{H}^1} = \|u\|_{\dot{H}^1}$ . For  $\dot{H}^1 \times L^2$  solution, the energy  $E(u(t), \partial_t u(t))$  and momentum  $M(u(t), \partial_t u(t))$  are conserved, where

$$E(u, v) = \frac{1}{2} \int v^2 + \frac{1}{2} \int |\nabla u|^2 - \frac{3}{10} \int |u|^{\frac{10}{3}}, \quad M(u, v) = \int v \nabla u.$$

Recall that the function  $W$  defined by

$$W(x) = \left(1 + \frac{|x|^2}{15}\right)^{-\frac{3}{2}}, \quad \Delta W + W^{\frac{7}{3}} = 0, \quad x \in \mathbb{R}^5, \quad (1.2)$$

is a stationary solution, called *soliton*, of (1.1). Using the Lorentz transformation on  $W$ , we obtain traveling solitons: for  $\ell \in \mathbb{R}^5$ , with  $|\ell| < 1$ , let

$$W_\ell(x) = W\left(\left(\frac{1}{\sqrt{1-|\ell|^2}} - 1\right) \frac{\ell(\ell \cdot x)}{|\ell|^2} + x\right); \quad (1.3)$$

then  $u(t, x) = \pm W_{\ell}(x - \ell t)$  is solution of (1.1).

Recall that an important conjecture in the field says that any global solution of (1.1) decomposes as  $t \rightarrow +\infty$  as a finite sum of (rescaled and translated) solitons plus a radiation (solution of the linear wave equation). Such a classification was achieved in the radial case in [8] (in space dimension 3) but is still widely open in the nonradial case (see [9] and references therein).

In this paper, we address the question of the construction of non trivial asymptotic behaviors in the nonradial case. In this context, multi-solitons are canonical objects behaving as  $t \rightarrow \infty$  exactly as the sum of several solitons in the energy space. The main result of this paper is the existence of 2-solitons for (1.1) and of  $K$ -solitons for  $K \geq 3$  for collinear speeds.

**Theorem 1** (Existence of multi-solitons). *Let  $K \geq 2$ . For  $k \in \{1, \dots, K\}$ , let  $\lambda_k^\infty > 0$ ,  $\mathbf{y}_k^\infty \in \mathbb{R}^5$ ,  $\iota_k = \pm 1$  and  $\ell_k \in \mathbb{R}^5$  with  $|\ell_k| < 1$ ,  $\ell_k \neq \ell_{k'}$  for  $k' \neq k$ . Assume that one of the following assumptions holds*

(A) Two-solitons ( $K = 2$ ).

(B) Collinear speeds. For all  $k \in \{1, \dots, K\}$ ,  $\ell_k = \ell_k \mathbf{e}_1$  where  $\ell_k \in (-1, 1)$ .

Then, there exist  $T_0 > 0$  and a solution  $u$  of (1.1) on  $[T_0, +\infty)$  in the energy space such that

$$\lim_{t \rightarrow +\infty} \left\| u(t) - \sum_{k=1}^K \frac{\iota_k}{(\lambda_k^\infty)^{3/2}} W_{\ell_k} \left( \frac{\cdot - \ell_k t - \mathbf{y}_k^\infty}{\lambda_k^\infty} \right) \right\|_{\dot{H}^1} = 0, \quad (1.4)$$

$$\lim_{t \rightarrow +\infty} \left\| \partial_t u(t) + \sum_{k=1}^K \frac{\iota_k}{(\lambda_k^\infty)^{5/2}} (\ell_k \cdot \nabla W_{\ell_k}) \left( \frac{\cdot - \ell_k t - \mathbf{y}_k^\infty}{\lambda_k^\infty} \right) \right\|_{L^2} = 0. \quad (1.5)$$

The question of existence and properties of multi-solitons for nonlinear models has a long history starting with the celebrated works of Fermi, Pasta and Ulam [10] and Kruskal and Zabusky [32], and closely related to the study of integrable equations by the inverse scattering transform. We refer in particular to the review work of Miura [24] on multi-solitons for the Korteweg-de Vries equation and to Zakharov and Shabat [33] for multi-solitons of the 1D cubic Schrödinger equation. Recall that in integrable cases, these solutions are very special: they are explicit and behave exactly as the sum of several solitons both at  $t \rightarrow +\infty$  and  $t \rightarrow -\infty$ . In particular, they describe the collision and interaction of several solitons globally in time, i.e. for all  $t \in (-\infty, +\infty)$ .

Apart from works on integrable models, there have been several proofs of existence of multi-solitons for nonlinear dispersive equations, starting with [22] for the  $L^2$  critical nonlinear Schrödinger equation and [17] for the subcritical and critical generalized Korteweg-de Vries equations. Note that [17] also contains a uniqueness result in the energy space, whose proof is specific to KdV type equations. Concerning existence, the general strategy of these works is to build backwards in time a sequence of approximate solutions satisfying uniform estimates and then to use a compactness argument. In [17] and also in [18], concerning the subcritical nonlinear Schrödinger equation, uniform estimates are deduced from long time stability arguments, adapted from the previous works [31] (for single solitons) and [20] (for several decoupled solitons). Later, the strategy of these works was extended to the case of exponentially unstable solitons, see [4] for the construction of multi-solitons and [3] for the classification of all multi-solitons of the supercritical generalized KdV equation. In these papers, the exponential instability is controled through a simple topological argument.

For the Klein-Gordon equation, the strategy was adapted by Cote and Munoz [5] (for real and unstable solitons) and Bellazzini, Ghimenti and Le Coz [1] (for complex, stable solitons). For the water-waves system, see the recent work of Ming, Rousset and Tzvetkov [23].

Note that all the works mentioned before are for exponentially decaying solitons, and thus exponentially small interactions as  $t \rightarrow +\infty$ . The main difficulty of constructing multi-solitons for (1.1) is due to the algebraic decay of  $W$ , which implies that the solitons have strong interactions, of order  $t^{-3}$ . For the Benjamin-Ono equation, multi-solitons exist with solitons behaving algebraically at  $\infty$ , but they are obtained explicitly using the integrability of the equation (see e.g. [21] and [25]). Stability and asymptotic stability of such multi-solitons is proved in [13], but relying on specific monotonicity formulas for KdV type equations. In [15], devoted to the construction of multi-solitons for the Hartree equation, solitons are also decaying algebraically. However, in that case, the potential related to the soliton is exponentially decaying, which allows a decoupling facilitating the construction of an approximate solution at order  $t^{-M}$  for arbitrarily large  $M$ . For  $M > M_0$  large enough, an actual solution can then be constructed close to this approximate solution. Such decoupling is not present in the case of the energy critical wave equation (1.1) and it seems delicate to construct sharp approximate multi-solitons (i.e. at order  $t^{-M}$  for large  $M$ ).

**1.2. Comments on Theorem 1.** (1) Each soliton being exponentially unstable, it can be derived as a consequence of the proof that the multi-solitons constructed in Theorem 1 are unstable. Uniqueness of multi-soliton in the energy space, up to the unstable directions, is an open problem as for the nonlinear Schrödinger equation. The uniqueness statements in [17] and [3] are specific to KdV-type equations.

The global behavior of  $u(t)$  i.e. for  $t < T_0$  is an open problem. We conjecture that it does not have the multi-soliton behavior as  $t \rightarrow -\infty$ . We refer to [19] for the proof of nonexistence of pure multi-solitons in the case of the (non integrable) quartic generalized Korteweg de Vries equation for a certain range of speeds.

(2) Dimension  $N \geq 6$ . We expect that Theorem 1 still holds true for the energy-critical wave equation for space dimensions  $N \geq 6$ . Indeed, at the formal level, all the important computations of this paper can be reproduced for  $N \geq 6$ . However, the lack of regularity of the nonlinearity create several additional technical difficulties, which we choose not to treat in this paper. Recall that such difficulties were overcome for the Cauchy problem in the energy space in [2].

(3) Dimension 3 and 4. We conjecture that in this case, there exists no multi-soliton in the sense (1.4)–(1.5), for any value of  $K \geq 2$ . Heuristically, from the asymptotics as  $|x| \rightarrow \infty$ ,  $W(x) \sim |x|^{2-N}$  in dimension  $N$ , the interaction between two solitons of different speeds is  $t^{2-N}$ , i.e.  $t^{-1}$  in dimension 3, and  $t^{-2}$  in dimension 4. Following our method, these interactions are too strong and create diverging terms in the construction. However, to prove nonexistence of multi-soliton rigorously, one would need *a priori* information on any multi-soliton, which is an open problem for any dimension  $N \geq 3$ .

**1.3. Strategy of the proof.** First, we note that Theorem 1 in case (A) follows from case (B) with  $K = 2$  and the Lorentz transformation. See Section 5 for a detailed proof, inspired by arguments in [14, 9].

The proof of Theorem 1 in case (B) follows the strategy by uniform estimates and compactness introduced in [17] and [18], but due to the algebraic decay of the solitons, proving

uniform estimates is more delicate. For  $k \in \{1, \dots, K\}$ , let  $\lambda_k^\infty > 0$ ,  $\mathbf{y}_k^\infty \in \mathbb{R}^5$  and  $\ell_k \in \mathbb{R}^5$  with  $|\ell_k| < 1$ ,  $\ell_k \neq \ell_{k'}$  for  $k' \neq k$ .

Let  $S_n \rightarrow +\infty$  as  $n \rightarrow \infty$  and, for each  $n$ , let  $u_n$  be the (backwards) solution of (1.1) with data at time  $S_n$

$$u_n(S_n, x) \sim \sum_{k=1}^K \frac{\iota_k}{(\lambda_k^\infty)^{3/2}} W_{\ell_k} \left( \frac{x - \ell_k S_n - \mathbf{y}_k^\infty}{\lambda_k^\infty} \right), \quad (1.6)$$

$$\partial_t u(S_n, x) \sim - \sum_{k=1}^K \frac{\iota_k}{(\lambda_k^\infty)^{5/2}} (\ell_k \cdot \nabla W_{\ell_k}) \left( \frac{x - \ell_k S_n - \mathbf{y}_k^\infty}{\lambda_k^\infty} \right). \quad (1.7)$$

(See (4.1) for a precise definition of  $(u_n(S_n), \partial_t u_n(S_n))$ ). The goal is to prove the following uniform estimates on the time interval  $[T_0, S_n]$ ,

$$\left\| u_n(t) - \sum_{k=1}^K \frac{\iota_k}{(\lambda_k^\infty)^{3/2}} W_{\ell_k} \left( \frac{\cdot - \ell_k t - \mathbf{y}_k^\infty}{\lambda_k^\infty} \right) \right\|_{\dot{H}^1} \lesssim \frac{1}{t}, \quad (1.8)$$

$$\left\| \partial_t u_n(t) + \sum_{k=1}^K \frac{\iota_k}{(\lambda_k^\infty)^{5/2}} \ell_k \cdot \nabla W_{\ell_k} \left( \frac{\cdot - \ell_k t - \mathbf{y}_k^\infty}{\lambda_k^\infty} \right) \right\|_{L^2} \lesssim \frac{1}{t}. \quad (1.9)$$

for  $T_0$  large independent of  $n$ . Indeed, the existence of a multi-soliton then follows easily from standard compactness arguments (note that we also obtain bounds on weighted higher order Sobolev norms for  $(u_n, \partial_t u_n)$  which facilitate the convergence). Thus, we now focus on the proof of (1.8)–(1.9). Note first that such long time stability estimates cannot be true for any initial data of the form (1.6)–(1.7); indeed, to take into account the exponential instability of each soliton  $W_{\ell_k}$ , we need to adjust the initial condition  $(u_n(S_n), \partial_t u_n(S_n))$ . This adjustment relies on a simple topological argument on  $K$  scalar parameters, first introduced in a similar context in [4].

We introduce

$$\varepsilon = u_n - \sum_k W_k, \quad \eta = \partial_t u_n + \sum_k (\ell_k \cdot \nabla W_k),$$

where

$$W_k(t, x) = \frac{\iota_k}{\lambda_k^{3/2}(t)} W_{\ell_k} \left( \frac{x - \ell_k t - \mathbf{y}_k(t)}{\lambda_k(t)} \right).$$

By a standard procedure, in the definition of  $W_k$ , the modulation parameters  $\lambda_k(t)$  and  $\mathbf{y}_k(t)$  are chosen close to  $\lambda_k^\infty$  and  $\mathbf{y}_k^\infty$  in order to obtain suitable orthogonality conditions on  $(\varepsilon, \eta)$ . The equation of  $(\varepsilon, \eta)$  is thus coupled by equations on  $\lambda_k$  and  $\mathbf{y}_k$ . See Lemma 3.1.

The general strategy of the proof of the uniform estimates (1.8)–(1.9) is to use global functionals that are locally of the form

$$\int_{x \sim \ell_k t + \mathbf{y}_k(t)} |\nabla \varepsilon|^2 + |\eta|^2 + 2(\ell_k \cdot \nabla \varepsilon) \eta - \frac{7}{3} |W_k|^{\frac{4}{3}} \varepsilon^2,$$

around each soliton  $W_k$ , i.e. in regions  $x \sim \ell_k t + \mathbf{y}_k(t)$ . Note that the coercivity of such functional under usual orthogonality conditions on  $(\varepsilon, \eta)$  is standard. The difficulty is to “glue” these  $K$  functionals to obtain a unique global functional on  $(\varepsilon, \eta)$  which is locally adapted to each soliton  $W_k$ .

In case (B) of Theorem 1, we assume  $\ell_k = \ell_k \mathbf{e}_1$  and  $-1 < \ell_1 < \dots < \ell_K < 1$ . To prove (1.8)-(1.9), we introduce the following energy functional

$$\mathcal{H}_K = \int \mathcal{E}_K + 2 \int (\chi_K(t, x) \partial_{x_1} \varepsilon) \eta,$$

where  $\mathcal{E}_K$  is the following “linearized energy density”

$$\mathcal{E}_K = |\nabla \varepsilon|^2 + |\eta|^2 - \frac{3}{5} \left( \left| \sum_k W_k + \varepsilon \right|^{\frac{10}{3}} - \left| \sum_k W_k \right|^{\frac{10}{3}} - \frac{10}{3} \left| \sum_k W_k \right|^{\frac{4}{3}} \left( \sum_k W_k \right) \varepsilon \right), \quad (1.10)$$

and the bounded function  $\chi_K(t, x)$  is equal to  $\ell_k$  in a neighborhood of the soliton  $W_k$  and close to  $\frac{x_1}{t}$  in “transition regions” between two solitons (see (4.15) for a precise definition). Note that the functional  $\mathcal{H}_K$  is inspired by the ones used in [17] and [18] for the construction of multi-solitons for (gKdV) and (NLS) equations in energy subcritical cases.

The functional  $\mathcal{H}_K$  has the following two important properties (see Proposition 4.2 for more precise statements):

(1)  $\mathcal{H}_K$  is coercive, in the sense that (up to unstable directions, to be controlled separately), it controls the size of  $(\varepsilon, \eta)$  in the energy space

$$\mathcal{H}_K \sim \|\varepsilon\|_{\dot{H}^1}^2 + \|\eta\|_{L^2}^2.$$

(2) The variation of  $\mathcal{H}_K$  is controled on  $[T_0, S_n]$  in the following (weak) sense

$$-\frac{d}{dt} (t^2 \mathcal{H}_K) \lesssim t^{-3}. \quad (1.11)$$

Note that the term  $t^{-3}$  in the right-hand side is related to interactions between solitons.

Therefore, integrating (1.11) on  $[t, S_n]$ , from (1.6)–(1.7), we find the uniform bound, for any  $t \in [T_0, S_n]$ ,

$$\|\varepsilon\|_{\dot{H}^1} + \|\eta\|_{L^2} \lesssim t^{-2}.$$

By time integration of the equations of the parameters, the above estimate implies

$$|\mathbf{y}_k(t) - \mathbf{y}_k^\infty| \lesssim t^{-1}, \quad |\lambda_k(t) - \lambda_k^\infty| \lesssim t^{-1},$$

and (1.8)–(1.9) follow.

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## 2. PRELIMINARIES

**2.1. Notation.** We denote

$$(g, \tilde{g})_{L^2} = \int g \tilde{g}, \quad \|g\|_{L^2}^2 = \int |g|^2, \quad (g, \tilde{g})_{\dot{H}^1} = \int \nabla g \cdot \nabla \tilde{g}, \quad \|g\|_{\dot{H}^1}^2 = \int |\nabla g|^2.$$

For

$$\vec{g} = \begin{pmatrix} g \\ h \end{pmatrix}, \quad \vec{\tilde{g}} = \begin{pmatrix} \tilde{g} \\ \tilde{h} \end{pmatrix},$$

set

$$\left( \vec{g}, \vec{\tilde{g}} \right)_{L^2} = (g, \tilde{g})_{L^2} + (h, \tilde{h})_{L^2}, \quad \left( \vec{g}, \vec{\tilde{g}} \right)_E = (g, \tilde{g})_{\dot{H}^1} + (h, \tilde{h})_{L^2}, \quad \|\vec{g}\|_E^2 = \|g\|_{\dot{H}^1}^2 + \|h\|_{L^2}^2.$$

When  $x_1$  is seen as a specific coordinate, denote

$$\bar{x} = (x_2, \dots, x_5), \quad \bar{\nabla}g = (\partial_{x_2}g, \dots, \partial_{x_5}g), \quad \bar{\Delta}g = \sum_{j=2}^5 \partial_{x_j}^2 g.$$

For  $-1 < \ell < 1$ ,

$$(g, \tilde{g})_{\dot{H}_\ell^1} = (1 - \ell^2) \int \partial_{x_1}g \partial_{x_1}\tilde{g} + \int \bar{\nabla}g \cdot \bar{\nabla}\tilde{g}, \quad \|g\|_{\dot{H}_\ell^1}^2 = (g, g)_{\dot{H}_\ell^1}$$

More generally, for  $\ell \in \mathbb{R}^5$  such that  $|\ell| < 1$ ,

$$(g, \tilde{g})_{\dot{H}_\ell^1} = \int [\nabla g \cdot \nabla \tilde{g} - (\ell \cdot \nabla g)(\ell \cdot \nabla \tilde{g})], \quad \|g\|_{\dot{H}_\ell^1}^2 = \|g\|_{\dot{H}^1}^2 - \|\ell \cdot \nabla g\|_{L^2}^2.$$

Observe that if we define

$$g_\ell(x) = g \left( \left( \frac{1}{\sqrt{1 - |\ell|^2}} - 1 \right) \frac{\ell(\ell \cdot x)}{|\ell|^2} + x \right),$$

and similarly  $\tilde{g}, \tilde{g}_\ell$ , then

$$(g_\ell, \tilde{g}_\ell)_{\dot{H}_\ell^1} = (1 - |\ell|^2)^{\frac{1}{2}} (g, \tilde{g})_{\dot{H}^1}. \quad (2.1)$$

Let  $\Lambda$  and  $\tilde{\Lambda}$  be the  $\dot{H}^1$  and  $L^2$  scaling operators defined as follows

$$\Lambda g = \frac{3}{2}g + x \cdot \nabla g, \quad \tilde{\Lambda}g = \frac{5}{2}g + x \cdot \nabla g, \quad \tilde{\Lambda}\nabla = \nabla\Lambda, \quad \tilde{\Lambda} = \begin{pmatrix} \tilde{\Lambda} \\ \Lambda \end{pmatrix}. \quad (2.2)$$

Let

$$J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Recall the Hardy and Sobolev inequalities, for any  $v \in \dot{H}^1$ ,

$$\int \frac{|v|^2}{|x|^2} \lesssim \int |\nabla v|^2, \quad (2.3)$$

$$\|v\|_{L^{10/3}} \lesssim \|\nabla v\|_{L^2}. \quad (2.4)$$

Set  $\langle x \rangle = (1 + |x|^2)^{\frac{1}{2}}$  and

$$\|v\|_{Y^0}^2 = \int (|v(x)|^2 + |\nabla v(x)|^2) \langle x \rangle dx, \quad \|v\|_{Y^1}^2 = \int (|\nabla v(x)|^2 + |\nabla^2 v(x)|^2) \langle x \rangle dx.$$

If  $g \in C([t_1, t_2], Y^0)$  then the unique solution  $v \in C([t_1, t_2], \dot{H}^1)$  of  $\partial_t^2 v - \Delta v = g$  with  $v(t_1) = 0$  and  $\partial_t v(t_1) = 0$ , satisfies  $(v, v_t) \in C([t_1, t_2], Y^1 \times Y^0)$  and

$$\|(v, v_t)(t)\|_{Y^1 \times Y^0} \leq \int_{t_1}^t \|g(s)\|_{Y^0} ds. \quad (2.5)$$

Moreover, the following estimate holds, for all  $v \in Y^1$ ,

$$\| |v|^{\frac{4}{3}} v \|_{Y^0} \lesssim \|v\|_{\dot{H}^1}^{\frac{1}{3}} \|v\|_{Y^1}^2. \quad (2.6)$$

Thus, it follows from a standard argument (fixed point) that (1.1) is locally well-posed in the space  $Y^1 \times Y^0$  with a time of existence depending only on the size of the  $Y^1 \times Y^0$  norm of the initial data.

For initial data in the energy space  $\dot{H}^1 \times L^2$ , the Cauchy problem is also locally well-posed in a certain sense, using suitable Strichartz estimates ; we refer to section 2 of [14] and references therein.

Denote

$$f(u) = |u|^{\frac{4}{3}}u, \quad F(u) = \frac{3}{10}|u|^{\frac{10}{3}}.$$

**2.2. Energy linearization around  $W$ .** Let

$$L = -\Delta - f'(W), \quad (Lg, g)_{L^2} = \int |\nabla g|^2 - f'(W)g^2,$$

$$H = \begin{pmatrix} L & 0 \\ 0 & \text{Id} \end{pmatrix}, \quad (H\vec{g}, \vec{g})_{L^2} = (Lg, g)_{L^2} + \|h\|_{L^2}^2.$$

Let  $\vec{g}$  be small in the energy space. Then, expanding, integrating by parts, using the equation of  $W$  and (2.4), one has

$$\begin{aligned} E(W + g, h) &= E(W, 0) - \int (\Delta W + f(W))g + \frac{1}{2} \left( \int |\nabla g|^2 - f'(W)g^2 \right) + \frac{1}{2} \int h^2 \\ &\quad - \int \left( F(W + g) - F(W) - f(W)g - \frac{1}{2}f'(W)g^2 \right) \\ &= E(W, 0) + \frac{1}{2} (Lg, g)_{L^2} + \frac{1}{2} \|h\|_{L^2}^2 + O(\|g\|_{\dot{H}^1}^3). \end{aligned} \quad (2.7)$$

In this paper addressing the case of several solitons, it is crucial to be able to spacially split the solitons. For some  $0 < \alpha \ll 1$  to be fixed, set

$$\varphi(x) = (1 + |x|^2)^{-\alpha} \quad (2.8)$$

We gather here some properties of the operator  $L$ .

**Lemma 2.1** (Spectral properties of  $L$ ). (i) Spectrum. *The operator  $L$  on  $L^2$  with domain  $H^2$  is a self-adjoint operator with essential spectrum  $[0, +\infty)$ , no positive eigenvalue and only one negative eigenvalue  $-\lambda_0$ , with a smooth radial positive eigenfunction  $Y \in \mathcal{S}(\mathbb{R}^5)$ . Moreover,*

$$L(\Lambda W) = L(\partial_{x_j} W) = 0, \quad \text{for any } j = 1, \dots, 5. \quad (2.9)$$

*There exists  $\mu > 0$  such that, for all  $g \in \dot{H}^1$ , the following holds.*

(ii) Coercivity with  $W$  orthogonality (Appendix D of [27]).

$$(Lg, g)_{L^2} \geq \mu \|g\|_{\dot{H}^1}^2 - \frac{1}{\mu} \left( (g, \Lambda W)_{\dot{H}^1}^2 + \sum_{j=1}^5 (g, \partial_{x_j} W)_{\dot{H}^1}^2 + (g, W)_{\dot{H}^1}^2 \right) \quad (2.10)$$

(iii) Coercivity with  $Y$  orthogonality.

$$(Lg, g)_{L^2} \geq \mu \|g\|_{\dot{H}^1}^2 - \frac{1}{\mu} \left( (g, \Lambda W)_{\dot{H}^1}^2 + \sum_{j=1}^5 (g, \partial_{x_j} W)_{\dot{H}^1}^2 + (g, Y)_{L^2}^2 \right) \quad (2.11)$$

(iv) Localized coercivity. *For  $\alpha > 0$  small enough,*

$$\int |\nabla g|^2 \varphi^2 - f'(W)g^2 \geq \mu \int |\nabla g|^2 \varphi^2 - \frac{1}{\mu} \left( (g, \Lambda W)_{\dot{H}^1}^2 + \sum_{j=1}^5 (g, \partial_{x_j} W)_{\dot{H}^1}^2 + (g, Y)_{L^2}^2 \right) \quad (2.12)$$

*Proof.* (i) contains well-known facts on  $L$  that are easily checked directly. We refer to Appendix D of [27] for the proof of (2.10). The proof of (iii) is standard since  $(LY, Y) < 0$ .

Proof of (2.12). By direct computations

$$\int |\nabla(g\varphi)|^2 = \int |\nabla g|^2 \varphi^2 - \int |g|^2 \varphi \Delta \varphi.$$

Note that (here the space dimension is 5)

$$\Delta \varphi = -2\alpha ((3 - 2\alpha)|x|^2 + 5) \frac{\varphi}{(1 + |x|^2)^2},$$

and thus  $|\Delta \varphi| \leq 10\alpha \frac{\varphi}{\langle x \rangle^2}$ , and thus by (2.3),

$$\int |g|^2 \varphi |\Delta \varphi| \leq 10\alpha \int |g|^2 \frac{\varphi^2}{\langle x \rangle^2} \leq \delta(\alpha) \int |\nabla(g\varphi)|^2,$$

where  $\delta(\alpha) \rightarrow 0$  as  $\alpha \rightarrow 0$ . This implies the following estimate

$$\left| \int |\nabla g|^2 \varphi^2 - \int |\nabla(g\varphi)|^2 \right| \leq \delta(\alpha) \int |\nabla(g\varphi)|^2. \quad (2.13)$$

We check that

$$|(g(1 - \varphi), \Lambda W)_{\dot{H}^1}| + |(g(1 - \varphi), \partial_{x_j} W)_{\dot{H}^1}| + |(g(1 - \varphi), Y)_{L^2}| \leq \delta(\alpha) \|g\varphi\|_{\dot{H}^1}. \quad (2.14)$$

Indeed, by the Cauchy-Schwarz inequality, the decay properties of  $W$  and Hardy inequality,

$$\begin{aligned} (g(1 - \varphi), \Lambda W)_{\dot{H}^1}^2 &= (g(1 - \varphi), \Delta(\Lambda W))_{L^2}^2 \\ &\leq \int \frac{(g\varphi)^2}{\langle x \rangle^2} \int |\Delta(\Lambda W)|^2 |1 - \varphi|^2 \frac{\langle x \rangle^2}{\varphi^2} \leq \delta(\alpha) \|g\varphi\|_{\dot{H}^1}^2; \end{aligned}$$

the rest of the proof of (2.14) is similar. We also have

$$\int W^{\frac{4}{3}} g^2 (1 - \varphi^2) \leq \left\| \frac{1 - \varphi^2}{\varphi^2} \langle x \rangle^2 W^{\frac{4}{3}} \right\|_{L^\infty} \int \frac{(g\varphi)^2}{\langle x \rangle^2} \lesssim \delta(\alpha) \|g\varphi\|_{\dot{H}^1}^2. \quad (2.15)$$

By (2.11) applied to  $g\varphi$  and then (2.14), for  $\alpha$  small,

$$\begin{aligned} (L(g\varphi), g\varphi)_{L^2} &\geq \mu \|g\varphi\|_{\dot{H}^1}^2 - \frac{1}{\mu} \left( (g\varphi, \Lambda W)_{\dot{H}^1}^2 + \sum_{j=1}^5 (g\varphi, \partial_{x_j} W)_{\dot{H}^1}^2 + (g\varphi, W)_{\dot{H}^1}^2 \right) \\ &\geq (\mu - \delta(\alpha)) \|g\varphi\|_{\dot{H}^1}^2 - \frac{1}{\mu} \left( (g, \Lambda W)_{\dot{H}^1}^2 + \sum_{j=1}^5 (g, \partial_{x_j} W)_{\dot{H}^1}^2 + (g, W)_{\dot{H}^1}^2 \right) \end{aligned}$$

Finally, using (2.13) and (2.15) we get (2.12), for  $\alpha$  small enough.  $\square$

**2.3. Energy linearization around  $W_\ell$ .** For  $-1 < \ell < 1$ , let

$$W_\ell(x) = W\left(\frac{x_1}{\sqrt{1 - \ell^2}}, \bar{x}\right), \quad (1 - \ell^2) \partial_{x_1}^2 W_\ell + \bar{\Delta} W_\ell + W_\ell^{\frac{7}{3}} = 0, \quad (2.16)$$

so that  $u(t, x) = W_\ell(x_1 - \ell t, \bar{x})$  is a solution of (1.1). Note that

$$E(W_\ell, -\ell \partial_{x_1} W_\ell) - \ell^2 \int |\partial_{x_1} W_\ell|^2 = (1 - \ell^2)^{\frac{1}{2}} E(W, 0). \quad (2.17)$$

Let

$$L_\ell = -(1 - \ell^2)\partial_{x_1}^2 - \overline{\Delta} - f'(W_\ell), \quad (2.18)$$

$$(L_\ell g, g)_{L^2} = (1 - \ell^2) \int |\partial_{x_1} g|^2 + \int (|\overline{\nabla} g|^2 - f'(W_\ell)g^2), \quad (2.19)$$

$$H_\ell = \begin{pmatrix} -\Delta - f'(W_\ell) & -\ell\partial_{x_1} \\ \ell\partial_{x_1} & \text{Id} \end{pmatrix}, \quad (H_\ell \vec{g}, \vec{g})_{L^2} = (L_\ell g, g)_{L^2} + \|\ell\partial_{x_1} g + h\|_{L^2}^2. \quad (2.20)$$

As before,  $L_\ell$  and  $H_\ell$  are related to the linearization of the energy around  $W_\ell$ . Indeed, proceeding as in (2.7),

$$\begin{aligned} & E(W_\ell + g, -\ell\partial_{x_1} W_\ell + h) + \ell \int \partial_{x_1}(W_\ell + g)(-\ell\partial_{x_1} W_\ell + h) \\ &= E(W_\ell, -\ell\partial_{x_1} W_\ell) - \ell^2 \int (\partial_{x_1} W_\ell)^2 \\ & - \int (\Delta W_\ell)g - \int f(W_\ell)g - \ell \int (\partial_{x_1} W_\ell)h + \ell^2 \int (\partial_{x_1}^2 W_\ell)g + \ell \int (\partial_{x_1} W_\ell)h \\ & + \frac{1}{2} \int |h|^2 + \frac{1}{2} \int (|\nabla g|^2 - f'(W_\ell)g^2) + \ell \int h\partial_{x_1} g + O(\|g\|_{H^1}^3). \end{aligned}$$

and thus, using (2.16) and (2.17),

$$\begin{aligned} & E(W_\ell + g, -\ell\partial_{x_1} W_\ell + h) + \ell \int \partial_{x_1}(W_\ell + g)(-\ell\partial_{x_1} W_\ell + h) \\ &= (1 - \ell^2)^{\frac{1}{2}} E(W, 0) + \frac{1}{2} (H_\ell \vec{g}, \vec{g})_{L^2} + O(\|g\|_{H^1}^3). \end{aligned}$$

The following functions appear when studying the properties of the operators  $H_\ell$  and  $H_\ell J$

$$\begin{aligned} \vec{Z}_\ell^\Lambda &= \begin{pmatrix} \Lambda W_\ell \\ -\ell\partial_{x_1} \Lambda W_\ell \end{pmatrix}, \quad \vec{Z}_\ell^{\nabla_j} = \begin{pmatrix} \partial_{x_j} W_\ell \\ -\ell\partial_{x_1} \partial_{x_j} W_\ell \end{pmatrix}, \quad \vec{Z}_\ell^W = \begin{pmatrix} W_\ell \\ -\ell\partial_{x_1} W_\ell \end{pmatrix}, \\ Y_\ell(x) &= Y\left(\frac{x_1}{\sqrt{1-\ell^2}}, \bar{x}\right), \quad \vec{Z}_\ell^\pm = \begin{pmatrix} \left(\ell\partial_{x_1} Y_\ell \pm \frac{\sqrt{\lambda_0}}{\sqrt{1-\ell^2}} Y_\ell\right) e^{\pm \frac{\ell\sqrt{\lambda_0}}{\sqrt{1-\ell^2}} x_1} \\ Y_\ell e^{\pm \frac{\ell\sqrt{\lambda_0}}{\sqrt{1-\ell^2}} x_1} \end{pmatrix}. \end{aligned}$$

We gather below several technical facts.

**Claim 1.** *The following hold for any  $-1 < \ell < 1$ ,*

(i) Properties of  $L_\ell$ .

$$L_\ell(\Lambda W_\ell) = L_\ell(\partial_{x_j} W_\ell) = 0, \quad L_\ell Y_\ell = -\lambda_0 Y_\ell, \quad L_\ell W_\ell = -\frac{4}{3} W_\ell^{\frac{7}{3}}, \quad (2.21)$$

(ii) Properties of  $H_\ell$  and  $H_\ell J$ .

$$H_\ell \vec{Z}_\ell^\Lambda = H_\ell \vec{Z}_\ell^{\nabla_j} = 0, \quad H_\ell \vec{Z}_\ell^W = -\frac{4}{3} \begin{pmatrix} W_\ell^{\frac{7}{3}} \\ 0 \end{pmatrix}, \quad (2.22)$$

$$\left( H_\ell \vec{Z}_\ell^W, \vec{Z}_\ell^W \right)_{L^2} = -\frac{4}{3} \int W_\ell^{\frac{10}{3}}, \quad -H_\ell J(\vec{Z}_\ell^\pm) = \pm \sqrt{\lambda_0} (1 - \ell^2)^{\frac{1}{2}} \vec{Z}_\ell^\pm, \quad (2.23)$$

$$\left( \vec{Z}_\ell^\Lambda, \vec{Z}_\ell^W \right)_E = \left( \vec{Z}_\ell^{\nabla_j}, \vec{Z}_\ell^W \right)_E = 0, \quad \left( \vec{Z}_\ell^\Lambda, \vec{Z}_\ell^\pm \right)_{L^2} = \left( \vec{Z}_\ell^{\nabla_j}, \vec{Z}_\ell^\pm \right)_{L^2} = 0. \quad (2.24)$$

(iii) Antecedents. *There exist  $\vec{z}_\ell^\pm$  such that*

$$H_\ell \vec{z}_\ell^\pm = \vec{Z}_\ell^\pm, \quad (H_\ell \vec{z}_\ell^\pm, \vec{z}_\ell^\pm)_{L^2} = 0, \quad \left( \vec{z}_\ell^\pm, \vec{Z}_\ell^\Lambda \right)_E = \left( \vec{z}_\ell^\pm, \vec{Z}_\ell^{\nabla_j} \right)_E = 0 \quad (2.25)$$

*Proof.* The proof of (2.21) follows from the same properties at  $\ell = 0$ .

Next, note that for any function  $g$ ,

$$H_\ell \begin{pmatrix} g \\ -\ell \partial_{x_1} g \end{pmatrix} = \begin{pmatrix} L_\ell g \\ 0 \end{pmatrix}, \quad \left( H_\ell \begin{pmatrix} g \\ -\ell \partial_{x_1} g \end{pmatrix}, \begin{pmatrix} g \\ -\ell \partial_{x_1} g \end{pmatrix} \right)_{L^2} = (L_\ell g, g)_{L^2}. \quad (2.26)$$

*Proof of (2.22).* First, by (2.26) and (2.21),  $H_\ell(\vec{Z}_\ell^\Lambda) = H_\ell(\vec{Z}_\ell^{\nabla_j}) = 0$ . The identity concerning  $\vec{Z}_\ell^W$  also follows directly from (2.26) and (2.21).

*Proof of (2.23).* Note that

$$-H_\ell J = \begin{pmatrix} -\ell \partial_{x_1} & \Delta + \frac{7}{3} W_\ell^{\frac{4}{3}} \\ \text{Id} & -\ell \partial_{x_1} \end{pmatrix}.$$

On the one hand,

$$\begin{aligned} & -\ell \partial_{x_1} \left( \left( \ell \partial_{x_1} Y_\ell \pm \frac{\sqrt{\lambda_0}}{\sqrt{1-\ell^2}} Y_\ell \right) e^{\pm \frac{\ell \sqrt{\lambda_0}}{\sqrt{1-\ell^2}} x_1} \right) + \Delta \left( Y_\ell e^{\pm \frac{\ell \sqrt{\lambda_0}}{\sqrt{1-\ell^2}} x_1} \right) + \frac{7}{3} W_\ell^{\frac{4}{3}} Y_\ell e^{\pm \frac{\ell \sqrt{\lambda_0}}{\sqrt{1-\ell^2}} x_1} \\ &= -(L_\ell Y_\ell) e^{\pm \frac{\ell \sqrt{\lambda_0}}{\sqrt{1-\ell^2}} x_1} \pm \frac{(1-\ell^2)\ell \sqrt{\lambda_0}}{\sqrt{1-\ell^2}} (\partial_{x_1} Y_\ell) e^{\pm \frac{\ell \sqrt{\lambda_0}}{\sqrt{1-\ell^2}} x_1} \\ &= \pm \sqrt{\lambda_0} (1-\ell^2)^{\frac{1}{2}} \left( \pm \sqrt{\lambda_0} (1-\ell^2)^{-\frac{1}{2}} Y_\ell + \ell (\partial_{x_1} Y_\ell) \right) e^{\pm \frac{\ell \sqrt{\lambda_0}}{\sqrt{1-\ell^2}} x_1}. \end{aligned}$$

On the other hand,

$$\left( \ell \partial_{x_1} Y_\ell \pm \frac{\sqrt{\lambda_0}}{\sqrt{1-\ell^2}} Y_\ell \right) e^{\pm \frac{\ell \sqrt{\lambda_0}}{\sqrt{1-\ell^2}} x_1} - \ell \partial_{x_1} \left( Y_\ell e^{\pm \frac{\ell \sqrt{\lambda_0}}{\sqrt{1-\ell^2}} x_1} \right) = \pm \sqrt{\lambda_0} (1-\ell^2)^{\frac{1}{2}} Y_\ell e^{\pm \frac{\ell \sqrt{\lambda_0}}{\sqrt{1-\ell^2}} x_1}$$

Thus,  $-H_\ell J(\vec{Z}_\ell^\pm) = \pm \sqrt{\lambda_0} (1-\ell^2)^{\frac{1}{2}} \vec{Z}_\ell^\pm$ .

*Proof of (2.24).* Since  $(\partial_{x_j} \Lambda W, \partial_{x_j} W)_{L^2} = 0$  ( $\dot{H}^1$  scaling) and  $(\partial_{x_j} \partial_{x_{j'}} W, \partial_{x_j} W)_{L^2} = 0$  (by parity), we have  $(\vec{Z}_\ell^\Lambda, \vec{Z}_\ell^W)_E = (\vec{Z}_\ell^{\nabla_k}, \vec{Z}_\ell^W)_E = 0$ . Next, from (2.24), the fact that  $H_\ell$  is self-adjoint in  $L^2$  and (2.22), we have

$$\mp \sqrt{\lambda_0} (1-\ell^2)^{\frac{1}{2}} (\vec{Z}_\ell^\Lambda, \vec{Z}_\ell^\pm)_{L^2} = (\vec{Z}_\ell^\Lambda, H_\ell J(\vec{Z}_\ell^\pm))_{L^2} = (H_\ell \vec{Z}_\ell^\Lambda, J(\vec{Z}_\ell^\pm))_{L^2} = 0.$$

The identity  $(\vec{Z}_\ell^{\nabla_j}, \vec{Z}_\ell^\pm)_{L^2} = 0$  is proved in a similar way.

*Proof of (2.25).* We set

$$\vec{z}_\ell^\pm = \mp \frac{J \vec{Z}_\ell^\pm}{\sqrt{\lambda_0} (1-\ell^2)^{1/2}} + \alpha^{\Lambda, \pm} \vec{Z}_\ell^\Lambda + \sum_{j=1}^5 \alpha^{\nabla_j, \pm} \vec{Z}_\ell^{\nabla_j},$$

where  $\alpha^{\Lambda, \pm}$  and  $\alpha^{\nabla_j, \pm}$  are chosen so that

$$(\vec{z}_\ell^\pm, \vec{Z}_\ell^\Lambda)_E = (\vec{z}_\ell^\pm, \vec{Z}_\ell^{\nabla_j})_E = 0.$$

By (2.22) and (2.23), we have  $H_\ell \vec{z}_\ell^\pm = \vec{Z}_\ell^\pm$ . Finally,  $H_\ell$  being self-adjoint, we have

$$(H_\ell \vec{z}_\ell^\pm, \vec{z}_\ell^\pm)_{L^2} = \mp \left( H_\ell \vec{z}_\ell^\pm, -\frac{J \vec{Z}_\ell^\pm}{\sqrt{\lambda_0}(1-\ell^2)} \right)_{L^2} = \mp \frac{1}{\sqrt{\lambda_0}(1-\ell^2)^{1/2}} (\vec{Z}_\ell^\pm, J \vec{Z}_\ell^\pm)_{L^2} = 0.$$

□

We claim the following coercivity results with  $\vec{Z}_k^\pm$  orthogonalities.

**Lemma 2.2.** *Let  $-1 < \ell < 1$ . There exists  $\mu > 0$  such that, for all  $\vec{g} \in \dot{H}^1 \times L^2$ , the following holds.*

(i) Coercivity of  $H_\ell$  with  $Z_\ell^\pm$  orthogonalities.

$$(H_\ell \vec{g}, \vec{g})_{L^2} \geq \mu \|\vec{g}\|_E^2 - \frac{1}{\mu} \left( (g, \Lambda W_\ell)_{\dot{H}_\ell^1}^2 + \sum_{j=1}^5 (g, \partial_{x_j} W_\ell)_{\dot{H}_\ell^1}^2 + (\vec{g}, \vec{Z}_\ell^+)_{L^2}^2 + (\vec{g}, \vec{Z}_\ell^-)_{L^2}^2 \right). \quad (2.27)$$

(ii) Localized coercivity. For  $\alpha > 0$  small enough,

$$\begin{aligned} & \int (|\nabla g|^2 \varphi^2 - f'(W_\ell) g^2 + h^2 \varphi^2 + 2\ell(\partial_{x_1} g) h \varphi^2) \\ & \geq \mu \int (|\nabla g|^2 + h^2) \varphi^2 - \frac{1}{\mu} \left( (g, \Lambda W_\ell)_{\dot{H}^1}^2 + \sum_{j=1}^5 (g, \partial_{x_j} W_\ell)_{\dot{H}^1}^2 + (\vec{g}, \vec{Z}_\ell^+)_{L^2}^2 + (\vec{g}, \vec{Z}_\ell^-)_{L^2}^2 \right). \end{aligned} \quad (2.28)$$

*Proof.* *Proof of (2.27).* By a standard argument, it is equivalent to prove

$$(g, \Lambda W_\ell)_{\dot{H}_\ell^1} = (g, \partial_{x_j} W_\ell)_{\dot{H}_\ell^1} = (\vec{g}, \vec{Z}_\ell^\pm)_{L^2} = 0 \quad \Rightarrow \quad (H_\ell \vec{g}, \vec{g})_{L^2} \geq \mu \|\vec{g}\|_E^2. \quad (2.29)$$

Note that the proof of (2.29) is largely inspired by Proposition 2 in [5], Lemma 5.1 in [7], and Proposition 5.5 in [6].

*Case  $\ell = 0$ .* Note that in this case  $\vec{Z}_0^\pm = \begin{pmatrix} \pm \sqrt{\lambda_0} Y \\ Y \end{pmatrix}$ , and  $g$  as in (2.29) thus satisfies the orthogonality conditions  $(g, \Lambda W)_{\dot{H}^1} = (g, \partial_{x_j} W)_{\dot{H}^1} = (g, Y)_{L^2} = 0$ . Then, (2.29) follows from (2.11).

*Case  $\ell \neq 0$ .* Note that (2.27) is thus equivalent to

$$(g, \Lambda W_\ell)_{\dot{H}_\ell^1} = (g, \partial_{x_j} W_\ell)_{\dot{H}_\ell^1} = (H_\ell \vec{g}, \vec{z}_\ell^\pm)_{L^2} = 0 \quad \Rightarrow \quad (H_\ell \vec{g}, \vec{g})_{L^2} \geq \mu_\ell \|\vec{g}\|_E^2. \quad (2.30)$$

We decompose  $g$  and  $\vec{z}_\ell^\pm$  as follows

$$\vec{g} = a \vec{Z}_\ell^W + \vec{g}^\perp, \quad \vec{z}_\ell^\pm = a^\pm \vec{Z}_\ell^W + \vec{z}_\ell^{\pm, \perp}, \quad \vec{g}^\perp = \begin{pmatrix} g^\perp \\ h^\perp \end{pmatrix}, \quad \vec{z}_\ell^{\pm, \perp} = \begin{pmatrix} z_{\ell,1}^{\pm, \perp} \\ z_{\ell,2}^{\pm, \perp} \end{pmatrix}, \quad (2.31)$$

where  $a$  and  $a^\pm$  are chosen so that

$$(g^\perp, W_\ell)_{\dot{H}_\ell^1} = (z_{\ell,1}^{\pm, \perp}, W_\ell)_{\dot{H}_\ell^1} = 0. \quad (2.32)$$

We still have

$$(g^\perp, \Lambda W_\ell)_{\dot{H}_\ell^1} = (g^\perp, \partial_{x_j} W_\ell)_{\dot{H}_\ell^1} = 0, \quad (\vec{z}_\ell^{\pm, \perp}, \Lambda W_\ell)_{\dot{H}_\ell^1} = (\vec{z}_\ell^{\pm, \perp}, \partial_{x_j} W_\ell)_{\dot{H}_\ell^1} = 0.$$

Note that since (see (2.22) and (2.16))

$$H_\ell \vec{Z}_\ell^W = -\frac{4}{3} \begin{pmatrix} W_\ell^{\frac{7}{3}} \\ 0 \end{pmatrix} = \frac{4}{3} \begin{pmatrix} (1 - \ell^2) \partial_{x_1}^2 W_\ell + \bar{\Delta} W_\ell \\ 0 \end{pmatrix},$$

(2.32) is equivalent to

$$\left( H_\ell \vec{g}^\perp, \vec{Z}_\ell^W \right)_{L^2} = \left( H_\ell \vec{z}_\ell^{\pm, \perp}, \vec{Z}_\ell^W \right)_{L^2} = 0. \quad (2.33)$$

The decompositions (2.31) being orthogonal with respect to  $(H_\ell, \cdot)_{L^2}$ , we have

$$\begin{aligned} (H_\ell \vec{g}, \vec{g})_{L^2} &= a^2 \left( H_\ell \vec{Z}_\ell^W, \vec{Z}_\ell^W \right)_{L^2} + \left( H_\ell \vec{g}^\perp, \vec{g}^\perp \right)_{L^2}, \\ 0 &= (H_\ell \vec{z}_\ell^\pm, \vec{z}_\ell^\pm)_{L^2} = (a^\pm)^2 \left( H_\ell \vec{Z}_\ell^W, \vec{Z}_\ell^W \right)_{L^2} + \left( H_\ell \vec{z}_\ell^{\pm, \perp}, \vec{z}_\ell^{\pm, \perp} \right)_{L^2}, \\ 0 &= (H_\ell \vec{g}, \vec{z}_\ell^\pm)_{L^2} = a a^\pm \left( H_\ell \vec{Z}_\ell^W, \vec{Z}_\ell^W \right)_{L^2} + \left( H_\ell \vec{g}^\perp, \vec{z}_\ell^{\pm, \perp} \right)_{L^2}, \end{aligned} \quad (2.34)$$

which imply (recall that  $\left( H_\ell \vec{Z}_\ell^W, \vec{Z}_\ell^W \right)_{L^2} < 0$ ), from (2.23),

$$(H_\ell \vec{g}, \vec{g})_{L^2} = - \frac{\left( H_\ell \vec{g}^\perp, \vec{z}_\ell^{\pm, \perp} \right)_{L^2} \left( H_\ell \vec{g}^\perp, \vec{z}_\ell^{\mp, \perp} \right)_{L^2}}{\sqrt{\left( H_\ell \vec{z}_\ell^{\mp, \perp}, \vec{z}_\ell^{\mp, \perp} \right)_{L^2} \left( H_\ell \vec{z}_\ell^{\pm, \perp}, \vec{z}_\ell^{\pm, \perp} \right)_{L^2}}} + \left( H_\ell \vec{g}^\perp, \vec{g}^\perp \right)_{L^2}. \quad (2.35)$$

Let

$$A = \sup_{\vec{\omega} \in \text{Span}(\vec{z}_\ell^{+, \perp}, \vec{z}_\ell^{-, \perp})} \left| \frac{\left( H_\ell \vec{\omega}, \vec{z}_\ell^{\mp, \perp} \right)_{L^2}}{\sqrt{\left( H_\ell \vec{z}_\ell^{\mp, \perp}, \vec{z}_\ell^{\mp, \perp} \right)_{L^2}}} \frac{\left( H_\ell \vec{\omega}, \vec{z}_\ell^{+, \perp} \right)_{L^2}}{\sqrt{\left( H_\ell \vec{z}_\ell^{+, \perp}, \vec{z}_\ell^{+, \perp} \right)_{L^2}}} \right|$$

Since  $(H_\ell, \cdot)$  is positive definite on  $\text{Span}(\Delta W_\ell, \Delta \Lambda W_\ell, \Delta \partial_{x_j} W_\ell)^\perp$ , applying Cauchy-Schwarz inequality to each of the term of the product above, we find  $A \leq 1$ . Moreover,  $A = 1$  would imply that  $\vec{z}_\ell^{\mp, \perp}$  and  $\vec{z}_\ell^{+, \perp}$  are proportional, which is clearly not true for  $\ell \neq 0$  (for example, due to different behavior at  $\infty$  of  $\vec{Z}_k^\pm$ ). Thus,  $A < 1$ . As a consequence, we also obtain that for all  $\vec{\omega} \in \text{Span}(\Delta W_\ell, \Delta \Lambda W_\ell, \Delta \partial_{x_j} W_\ell)^\perp$ ,

$$\left| \frac{\left( H_\ell \vec{\omega}, \vec{z}_\ell^{\mp, \perp} \right)_{L^2}}{\sqrt{\left( H_\ell \vec{z}_\ell^{\mp, \perp}, \vec{z}_\ell^{\mp, \perp} \right)_{L^2}}} \frac{\left( H_\ell \vec{\omega}, \vec{z}_\ell^{+, \perp} \right)_{L^2}}{\sqrt{\left( H_\ell \vec{z}_\ell^{+, \perp}, \vec{z}_\ell^{+, \perp} \right)_{L^2}}} \right| \leq A (H_\ell \vec{\omega}, \vec{\omega})_{L^2}.$$

Thus, by (2.35) and then (2.10) (after change of variables),

$$(H_\ell \vec{g}, \vec{g})_{L^2} \geq (1 - A) \left( H_\ell \vec{g}^\perp, \vec{g}^\perp \right)_{L^2} \geq c \left\| \vec{g}^\perp \right\|_E^2.$$

The result then follows from  $|a| \lesssim \left\| \vec{g}^\perp \right\|_E$  from (2.34).

Proof of (2.28). First, we apply (2.27) on  $\vec{g}\varphi$ :

$$(H_\ell(\vec{g}\varphi), \vec{g}\varphi)_{L^2} \geq \mu \|\vec{g}\varphi\|_E^2 - \frac{1}{\mu} \left( (\vec{g}\varphi, \Lambda W_\ell)_{\dot{H}_\ell^1}^2 + \sum_{j=1}^5 (\vec{g}\varphi, \partial_{x_j} W_\ell)_{\dot{H}_\ell^1}^2 + (\vec{g}\varphi, \vec{Z}_\ell^+)_{L^2}^2 + (\vec{g}\varphi, \vec{Z}_\ell^-)_{L^2}^2 \right).$$

Recall that

$$(H_\ell(\vec{g}\varphi), \vec{g}\varphi)_{L^2} = \int |\nabla(g\varphi)|^2 - \frac{7}{3} \int W_\ell^{\frac{4}{3}} g^2 \varphi^2 + 2\ell \int \partial_{x_1}(g\varphi)(h\varphi) + \int h^2 \varphi^2.$$

Note that  $\partial_{x_1}\varphi = \frac{-2\alpha x_1}{1+|x|^2}\varphi$  and so

$$\begin{aligned} \left| \int \partial_{x_1}(g\varphi)(g\varphi) - \int (\partial_{x_1}g)h\varphi^2 \right| &= \left| \int gh(\partial_{x_1}\varphi)\varphi \right| \leq C\alpha \int |g||h| \frac{\varphi^2}{\langle x \rangle} \\ &\leq C\alpha \left( \int \frac{(g\varphi)^2}{\langle x \rangle^2} \right)^{\frac{1}{2}} \left( \int |h|^2 \varphi^2 \right)^{\frac{1}{2}} \leq C\alpha \int |\nabla(g\varphi)|^2 + C\alpha \int |h|^2 \varphi^2. \end{aligned}$$

Thus, using (2.13),

$$\left| (H_\ell(\vec{g}\varphi), \vec{g}\varphi)_{L^2} - \int (|\nabla g|^2 - f'(W_\ell)g^2 + h^2 + 2\ell(\partial_{x_1}g)h) \varphi^2 \right| \leq \delta(\alpha) \|\vec{g}\varphi\|_E^2.$$

To complete the proof, we just notice that as in (2.14)

$$\left| (\vec{g}(1-\varphi), \vec{Z}_\ell^\pm)_{L^2} \right| \leq \delta(\alpha) \|\vec{g}\varphi\|_E, \quad (2.36)$$

and similarly for the other scalar products appearing in (2.28), and as in (2.15),

$$\int W_\ell^{\frac{4}{3}} g^2 (1-\varphi^2) \lesssim \delta(\alpha) \|g\varphi\|_{\dot{H}^1}^2. \quad (2.37)$$

Combining these estimates, we obtain (2.28), for  $\alpha$  small enough.  $\square$

**2.4. Energy linearization around  $W_\ell$ .** We only define some notation generalizing the previous section. For  $\ell \in \mathbb{R}^5$  such that  $|\ell| < 1$ ,  $W_\ell$  defined in (1.3) solves

$$\Delta W_\ell - \ell \cdot \nabla(\ell \cdot \nabla W_\ell) + W_\ell^{\frac{7}{3}} = 0. \quad (2.38)$$

The following operators are related to the linearization of the energy around  $W_\ell$

$$L_\ell = -\Delta - \ell \cdot \nabla(\ell \cdot \nabla) - f'(W_\ell), \quad H_\ell = \begin{pmatrix} -\Delta - f'(W_\ell) & -\ell \cdot \nabla \\ \ell \cdot \nabla & \text{Id} \end{pmatrix}.$$

Set

$$\begin{aligned} \vec{Z}_\ell^\Lambda &= \begin{pmatrix} \Lambda W_\ell \\ -\ell \cdot \nabla(\Lambda W_\ell) \end{pmatrix}, \quad \vec{Z}_\ell^{\nabla_j} = \begin{pmatrix} \partial_{x_j} W_\ell \\ -\ell \cdot \nabla(\partial_{x_j} W_\ell) \end{pmatrix}, \quad \vec{Z}_\ell^W = \begin{pmatrix} W_\ell \\ -\ell \cdot \nabla W_\ell \end{pmatrix}, \\ Y_\ell &= Y \left( \left( \frac{1}{\sqrt{1-|\ell|^2}} - 1 \right) \frac{\ell(\ell \cdot x)}{|\ell|^2} + x \right), \quad \vec{Z}_\ell^\pm = \begin{pmatrix} \left( \ell \cdot \nabla Y_\ell \pm \frac{\sqrt{\lambda_0}}{\sqrt{1-|\ell|^2}} Y_\ell \right) e^{\pm \frac{\sqrt{\lambda_0}}{\sqrt{1-|\ell|^2}} \ell \cdot x} \\ Y_\ell e^{\pm \frac{\sqrt{\lambda_0}}{\sqrt{1-|\ell|^2}} \ell \cdot x} \end{pmatrix}. \end{aligned}$$

Note from (2.24) and (2.23),

$$(\vec{Z}_\ell^\Lambda, \vec{Z}_\ell^\pm)_{L^2} = (\vec{Z}_\ell^{\nabla_j}, \vec{Z}_\ell^\pm)_{L^2} = 0, \quad (2.39)$$

$$-H_\ell J \vec{Z}_\ell^\pm = \pm \sqrt{\lambda_0} (1 - |\ell|^2)^{\frac{1}{2}} \vec{Z}_\ell^\pm. \quad (2.40)$$

### 3. DECOMPOSITION AROUND THE SUM OF $K$ SOLITONS

We prove in this section a general decomposition around  $K$  solitons. Let  $K \geq 1$  and for any  $k \in \{1, \dots, K\}$ , let  $\lambda_k^\infty > 0$ ,  $\mathbf{y}_k^\infty \in \mathbb{R}^5$ ,  $\ell_k \in \mathbb{R}^5$ ,  $|\ell_k| < 1$  with  $\ell_{k'} \neq \ell_k$  for  $k' \neq k$ .

First, for  $\vec{G} = (G, H)$ , set

$$(\theta_k^\infty G)(t, x) = \frac{\iota_k}{(\lambda_k^\infty)^{3/2}} G\left(\frac{x - \ell_k t - \mathbf{y}_k^\infty}{\lambda_k^\infty}\right), \quad \vec{\theta}_k^\infty \vec{G} = \begin{pmatrix} \theta_k^\infty G \\ \frac{\theta_k^\infty}{\lambda_k^\infty} H \end{pmatrix}.$$

In particular, set

$$W_k^\infty = \theta_k^\infty W_{\ell_k}, \quad \vec{W}_k^\infty = \begin{pmatrix} \theta_k^\infty W_{\ell_k} \\ -\frac{\ell_k}{\lambda_k^\infty} \cdot \theta_k^\infty (\nabla W_{\ell_k}) \end{pmatrix}.$$

Second, for  $C^1$  functions  $\lambda_k(t) > 0$ ,  $\mathbf{y}_k(t) \in \mathbb{R}^5$  to be chosen, let

$$(\theta_k G)(t, x) = \frac{\iota_k}{\lambda_k^{3/2}(t)} G\left(\frac{x - \ell_k t - \mathbf{y}_k(t)}{\lambda_k(t)}\right), \quad \vec{\theta}_k \vec{G} = \begin{pmatrix} \theta_k G \\ \frac{\theta_k}{\lambda_k} H \end{pmatrix}, \quad \vec{\theta}_k \vec{G} = \begin{pmatrix} \frac{\theta_k}{\lambda_k} G \\ \theta_k H \end{pmatrix}. \quad (3.1)$$

In particular, set

$$W_k = \theta_k W_{\ell_k}, \quad \vec{W}_k = \begin{pmatrix} \theta_k W_{\ell_k} \\ -\frac{\ell_k}{\lambda_k} \cdot \theta_k (\nabla W_{\ell_k}) \end{pmatrix}. \quad (3.2)$$

In what follows  $\sum_{k=1}^K$  is often simply denoted by  $\sum_k$ .

**Lemma 3.1** (Properties of the decomposition). *There exist  $T_0 \gg 1$  and  $0 < \delta_0 \ll 1$  such that if  $u(t)$  is a solution of (1.1) on  $[T_1, T_2]$ , where  $T_0 \leq T_1 < T_2$ , such that*

$$\forall t \in [T_1, T_2], \quad \left\| \vec{u}(t) - \sum_k \vec{W}_k^\infty(t) \right\|_{\dot{H}^1 \times L^2} \leq \delta_0, \quad (3.3)$$

*then there exist  $C^1$  functions  $\lambda_k > 0$ ,  $\mathbf{y}_k$  on  $[T_1, T_2]$  such that,  $\vec{\varepsilon}(t)$  being defined by*

$$\vec{\varepsilon} = \begin{pmatrix} \varepsilon \\ \eta \end{pmatrix}, \quad \vec{u} = \begin{pmatrix} u \\ u_t \end{pmatrix} = \sum_k \vec{W}_k + \vec{\varepsilon}, \quad (3.4)$$

*the following hold on  $[T_1, T_2]$ .*

(i) First properties of the decomposition.

$$(\varepsilon, \theta_k (\Lambda W_{\ell_k}))_{\dot{H}_{\ell_k}^1} = (\varepsilon, \theta_k (\partial_{x_j} W_{\ell_k}))_{\dot{H}_{\ell_k}^1} = 0, \quad (3.5)$$

$$|\lambda_k(t) - \lambda_k^\infty| + |\mathbf{y}_k(t) - \mathbf{y}_k^\infty| + \|\vec{\varepsilon}\|_E \lesssim \left\| \vec{u}(t) - \sum_k \vec{W}_k^\infty(t) \right\|_{\dot{H}^1 \times L^2} \quad (3.6)$$

(ii) Equation of  $\vec{\varepsilon}$ .

$$\begin{cases} \varepsilon_t = \eta + \text{Mod}_\varepsilon \\ \eta_t = \Delta\varepsilon + f\left(\sum_k W_k + \varepsilon\right) - f\left(\sum_k W_k\right) + R_W + \text{Mod}_\eta \end{cases} \quad (3.7)$$

where

$$R_W = f\left(\sum_k W_k\right) - \sum_k f(W_k), \quad (3.8)$$

$$\text{Mod}_\varepsilon = \sum_k \frac{\dot{\lambda}_k}{\lambda_k} \theta_k(\Lambda W_{\ell_k}) + \sum_k \frac{\dot{\mathbf{y}}_k}{\lambda_k} \cdot \theta_k(\nabla W_{\ell_k}) \quad (3.9)$$

$$\text{Mod}_\eta = - \sum_k \frac{\dot{\lambda}_k}{\lambda_k^2} \ell_k \cdot \theta_k(\nabla \Lambda W_{\ell_k}) - \sum_k \frac{\dot{\mathbf{y}}_k}{\lambda_k^2} \cdot \theta_k(\nabla(\ell_k \cdot \nabla W_{\ell_k})). \quad (3.10)$$

(iii) Parameters equations.

$$\sum_k |\dot{\lambda}_k(t)| + |\dot{\mathbf{y}}_k(t)| \lesssim \|\vec{\varepsilon}(t)\|_E. \quad (3.11)$$

(iv) Unstable directions. *Let*

$$z_k^\pm(t) = \left( \vec{\varepsilon}(t), \vec{\theta}_k \vec{Z}_{\ell_k}^\pm \right)_{L^2}. \quad (3.12)$$

Then,

$$\left| \frac{d}{dt} z_k^\pm(t) \mp \frac{\sqrt{\lambda_0}}{\lambda_k} (1 - |\ell_k|^2)^{\frac{1}{2}} z_k^\pm(t) \right| \lesssim \|\vec{\varepsilon}(t)\|_E^2 + \frac{\|\vec{\varepsilon}(t)\|_E}{t} + \frac{1}{t^3}. \quad (3.13)$$

*Proof. Step 1.* Decomposition. Let  $T_0 \gg 1$ , fix  $t \geq T_0$  and assume that (3.3) holds for  $t$ . Let

$$\Gamma^\infty = (\lambda_k^\infty, \mathbf{y}_k^\infty)_{k \in \{1, \dots, K\}}, \quad \Gamma = (\lambda_k, \mathbf{y}_k)_{k \in \{1, \dots, K\}} \in ((0, +\infty) \times \mathbb{R}^5)^K,$$

where  $\lambda_k$  and  $\mathbf{y}_k$  are to be found (depending on  $t$ ). Consider the map

$$\begin{aligned} \Phi : \dot{H}^1 \times ((0, +\infty) \times \mathbb{R}^5)^K &\rightarrow \mathbb{R}^{6K} \\ (\omega, \Gamma) &\mapsto \left( \left( \omega + \sum_{k'} W_{k'}^\infty - \sum_{k'} \theta_{k'} W_{\ell_{k'}}, \theta_k(\Lambda W_{\ell_k}) \right)_{\dot{H}_{\ell_k}^1}, \right. \\ &\quad \left( \omega + \sum_{k'} W_{k'}^\infty - \sum_{k'} \theta_{k'} W_{\ell_{k'}}, \theta_k(\partial_{x_1} W_{\ell_k}) \right)_{\dot{H}_{\ell_k}^1}, \\ &\quad \dots, \left( \omega + \sum_{k'} W_{k'}^\infty - \sum_{k'} \theta_{k'} W_{\ell_{k'}}, \theta_k(\partial_{x_5} W_{\ell_k}) \right)_{\dot{H}_{\ell_k}^1} \right)_{k \in \{1, \dots, K\}}, \end{aligned}$$

where  $\theta_k$  is defined in (3.1). By explicit computations, we have

$$\begin{aligned} & \left( d_\Gamma \Phi(0, \Gamma^\infty) \cdot \tilde{\Gamma} \right)_k \\ &= \left( \sum_{k'} \frac{\tilde{\lambda}_{k'}}{\lambda_{k'}^\infty} (\theta_{k'}^\infty(\Lambda W_{\ell_{k'}}), \theta_k^\infty(\Lambda W_{\ell_k}))_{\dot{H}_{\ell_k}^1} + \sum_{k'} \frac{\tilde{\mathbf{y}}_{k'}}{\lambda_{k'}^\infty} \cdot (\theta_{k'}^\infty(\nabla W_{\ell_{k'}}), \theta_k^\infty(\Lambda W_{\ell_k}))_{\dot{H}_{\ell_k}^1}, \right. \\ & \quad \sum_{k'} \frac{\tilde{\lambda}_{k'}}{\lambda_{k'}^\infty} (\theta_{k'}^\infty(\Lambda W_{\ell_{k'}}), \theta_k^\infty(\partial_{x_1} W_{\ell_k}))_{\dot{H}_{\ell_k}^1} + \sum_{k'} \frac{\tilde{\mathbf{y}}_{k'}}{\lambda_{k'}^\infty} \cdot (\theta_{k'}^\infty(\nabla W_{\ell_{k'}}), \theta_k^\infty(\partial_{x_1} W_{\ell_k}))_{\dot{H}_{\ell_k}^1}, \dots, \\ & \quad \left. \sum_{k'} \frac{\tilde{\lambda}_{k'}}{\lambda_{k'}^\infty} (\theta_{k'}^\infty(\Lambda W_{\ell_{k'}}), \theta_k^\infty(\partial_{x_5} W_{\ell_k}))_{\dot{H}_{\ell_k}^1} + \sum_{k'} \frac{\tilde{\mathbf{y}}_{k'}}{\lambda_{k'}^\infty} \cdot (\theta_{k'}^\infty(\nabla W_{\ell_{k'}}), \theta_k^\infty(\partial_{x_5} W_{\ell_k}))_{\dot{H}_{\ell_k}^1} \right), \end{aligned}$$

Thus, by parity property,  $(\partial_{x_j} W, \partial_{x'_j} W)_{\dot{H}^1} = 0$  and the decay properties of  $W$ ,

$$\begin{aligned} & \left( d_\Gamma \Phi(0, \Gamma^\infty) \cdot \tilde{\Gamma} \right)_k \\ &= \left( \frac{\tilde{\lambda}_k}{\lambda_k^\infty} (\theta_k^\infty(\Lambda W_{\ell_k}), \theta_k^\infty(\Lambda W_{\ell_k}))_{\dot{H}_{\ell_k}^1}, \frac{\tilde{\mathbf{y}}_{k,1}}{\lambda_k^\infty} (\theta_k^\infty(\partial_{x_1} W_{\ell_k}), \theta_k^\infty(\partial_{x_1} W_{\ell_k}))_{\dot{H}_{\ell_k}^1}, \dots, \right. \\ & \quad \left. \sum_k \frac{\tilde{\mathbf{y}}_{k,5}}{\lambda_k^\infty} (\theta_k^\infty(\partial_{x_5} W_{\ell_k}), \theta_k^\infty(\partial_{x_5} W_{\ell_k}))_{\dot{H}_{\ell_k}^1} \right) + \mathcal{E} \cdot \tilde{\Gamma}, \end{aligned}$$

where  $\|\mathcal{E}\| \lesssim \frac{1}{T_0}$ . Hence,  $d_\Gamma \Phi(0, \Gamma^\infty)$  is invertible for  $T_0$  large enough, with a lower bound uniform in  $\Gamma^\infty$ . Moreover,  $\Phi(0, \Gamma^\infty) = 0$ . Therefore, by the implicit function theorem (in fact, a uniform variant of the IFT), there exist  $0 < \delta_1 \ll 1$ ,  $0 < \delta_2 \ll 1$ , and a continuous map

$$\Psi : B_{\dot{H}^1}(0, \delta_1) \rightarrow B_{((0, +\infty) \times \mathbb{R}^5)^K}(\Gamma^\infty, \delta_2),$$

such that for all  $\omega \in B_{\dot{H}^1}(0, \delta_1)$  and all  $\Gamma \in B_{((0, +\infty) \times \mathbb{R}^5)^K}(\Gamma^\infty, \delta_2)$ ,

$$\Phi(\omega, \Gamma) = 0 \quad \text{if and only if} \quad \Gamma = \Psi(\omega).$$

Moreover,

$$|\Psi(\omega) - \Gamma^\infty| \lesssim \|\omega\|_{\dot{H}^1}.$$

This defines a continuous map  $t \in [T_1, T_2] \mapsto (\lambda_k(t), \mathbf{y}_k(t))_{k \in \{1, \dots, K\}}$  such that

$$|\lambda_k(t) - \lambda_k^\infty| + |\mathbf{y}_k(t) - \mathbf{y}_k^\infty| \lesssim \left\| u(t) - \sum_k W_k^\infty(t) \right\|_{\dot{H}^1}$$

and such that  $\vec{\varepsilon}(t)$  defined by (3.4) satisfies the orthogonality conditions (3.5). Since

$$\|W_k(t) - W_k^\infty(t)\|_{\dot{H}^1} \lesssim |\lambda_k(t) - \lambda_k^\infty| + |\mathbf{y}_k(t) - \mathbf{y}_k^\infty|, \quad (3.14)$$

we have

$$\begin{aligned} \|\vec{\varepsilon}(t)\|_E &\lesssim \sum_k \|W_k(t) - W_k^\infty(t)\|_{\dot{H}^1} + \left\| \vec{u}(t) - \sum_k \vec{W}_k^\infty(t) \right\|_{\dot{H}^1 \times L^2} \\ &\lesssim \left\| \vec{u}(t) - \sum_k \vec{W}_k^\infty(t) \right\|_{\dot{H}^1 \times L^2}, \end{aligned}$$

and (3.6) is proved.

For future reference, note that

$$\left\| \langle x \rangle^{1/2} \nabla (W_k(t) - W_k^\infty(t)) \right\|_{L^2} \lesssim t^{\frac{1}{2}} (|\lambda_k(t) - \lambda_k^\infty| + |\mathbf{y}_k(t) - \mathbf{y}_k^\infty|). \quad (3.15)$$

Thus, if  $(u(t), \partial_t u(t)) \in Y^1 \times Y^0$ , then we have

$$\begin{aligned} & \left\| \langle x \rangle^{1/2} \nabla \varepsilon(t) \right\|_{L^2} + \left\| \langle x \rangle^{1/2} \eta(t) \right\|_{L^2} \lesssim \left\| \langle x \rangle^{1/2} \nabla \left( u(t) - \sum_k W_k^\infty(t) \right) \right\|_{L^2} \\ & + \left\| \langle x \rangle^{1/2} \left( \partial_t u(t) + \sum_k (\ell_k \cdot \nabla W_k^\infty(t)) \right) \right\|_{L^2} + t^{\frac{1}{2}} \left\| \vec{u}(t) - \sum_k \vec{W}_k^\infty(t) \right\|_{H^1 \times L^2}, \end{aligned}$$

and also

$$\|\vec{\varepsilon}(t)\|_{Y^1 \times Y^0} \lesssim t^{\frac{1}{2}} \left\| \vec{u}(t) - \sum_k \vec{W}_k^\infty(t) \right\|_{Y^1 \times Y^0}. \quad (3.16)$$

**Step 2.** Equation of  $\vec{\varepsilon}$  and parameter estimates. We formally derive the equations of  $\vec{\varepsilon}(t)$ ,  $\lambda_k(t)$  and  $\mathbf{y}_k(t)$  from the equation of  $u$ . First,

$$\begin{aligned} \varepsilon_t &= u_t - \sum_k \partial_t W_k = \eta - \sum_k \frac{\ell_k}{\lambda_k} \cdot \theta_k(\nabla W_{\ell_k}) - \sum_k \partial_t(\theta_k W_{\ell_k}) \\ &= \eta + \sum_k \frac{\dot{\lambda}_k}{\lambda_k} \theta_k(\Lambda W_{\ell_k}) + \sum_k \frac{\dot{\mathbf{y}}_k}{\lambda_k} \cdot \theta_k(\nabla W_{\ell_k}), \end{aligned} \quad (3.17)$$

since, by direct computations,

$$\partial_t(\theta_k W_{\ell_k}) = -\frac{\ell_k}{\lambda_k} \cdot \theta_k(\nabla W_{\ell_k}) - \frac{\dot{\lambda}_k}{\lambda_k} \theta_k(\Lambda W_{\ell_k}) - \frac{\dot{\mathbf{y}}_k}{\lambda_k} \cdot \theta_k(\nabla W_{\ell_k}). \quad (3.18)$$

Second (using (2.2))

$$\begin{aligned} \eta_t &= u_{tt} + \partial_t \left( \sum_k \frac{\ell_k}{\lambda_k} \cdot \theta_k(\nabla W_{\ell_k}) \right) = \Delta u + |u|^{\frac{4}{3}} u - \sum_k \frac{\ell_k}{\lambda_k^2} \cdot \theta_k(\nabla(\ell_k \cdot \nabla W_{\ell_k})) \\ &\quad - \sum_k \frac{\dot{\lambda}_k}{\lambda_k^2} \ell_k \cdot \theta_k(\nabla \Lambda W_{\ell_k}) - \sum_k \frac{\dot{\mathbf{y}}_k}{\lambda_k^2} \cdot \theta_k(\nabla(\ell_k \cdot \nabla W_{\ell_k})). \end{aligned}$$

Using  $u = \sum_k \theta_k W_{\ell_k} + \varepsilon$ , we have

$$\Delta u = \sum_k \frac{\theta_k}{\lambda_k^2} (\Delta W_{\ell_k}) + \Delta \varepsilon, \quad (3.19)$$

and

$$|u|^{\frac{4}{3}} u = f(u) = \sum_k f(\theta_k W_{\ell_k}) + \left( \sum_k f'(\theta_k W_{\ell_k}) \right) \varepsilon + R_{\text{NL}} + R_W, \quad (3.20)$$

where  $R_W$  is defined in (3.8) and

$$R_{\text{NL}} = f \left( \sum_k W_k + \varepsilon \right) - f \left( \sum_k W_k \right) - f' \left( \sum_k W_k \right) \varepsilon$$

Since

$$f(\theta_k W_{\ell_k}) = \frac{\theta_k}{\lambda_k^2} f(W_{\ell_k}), \quad f'(\theta_k W_{\ell_k}) = \frac{\theta_k}{\lambda_k^{\frac{1}{2}}} f'(W_{\ell_k}),$$

we obtain

$$\Delta u + |u|^{\frac{4}{3}} u = \sum_k \frac{\theta_k}{\lambda_k^2} \left( \Delta W_{\ell_k} + W_{\ell_k}^{\frac{7}{3}} \right) + \Delta \varepsilon + \frac{7}{3} \left( \sum_k \frac{\theta_k}{\lambda_k^{\frac{1}{2}}} W_{\ell_k}^{\frac{4}{3}} \right) \varepsilon + R_{\text{NL}} + R_W.$$

Using (2.38), we obtain

$$\begin{aligned} \eta_t &= \Delta \varepsilon + \frac{7}{3} \left( \sum_k \frac{\theta_k}{\lambda_k^{\frac{1}{2}}} W_{\ell_k}^{\frac{4}{3}} \right) \varepsilon + R_{\text{NL}} + R_W \\ &\quad - \sum_k \dot{\lambda}_k \frac{\ell_k}{\lambda_k^2} \cdot \theta_k (\nabla \Lambda W_{\ell_k}) - \sum_k \frac{\dot{\mathbf{y}}_k}{\lambda_k^2} \cdot \theta_k (\nabla (\ell_k \cdot \nabla W_{\ell_k})). \end{aligned}$$

In conclusion for  $\vec{\varepsilon}$ , we obtain

$$\vec{\varepsilon}_t = \vec{\mathcal{L}} \vec{\varepsilon} + \vec{\text{Mod}} + \vec{R}_{\text{NL}} + \vec{R}_W, \quad (3.21)$$

where

$$\vec{\mathcal{L}} = \begin{pmatrix} 0 & 1 \\ \Delta + \frac{7}{3} \left( \sum_k \frac{\theta_k}{\lambda_k^{\frac{1}{2}}} W_{\ell_k}^{\frac{4}{3}} \right) & 0 \end{pmatrix}, \quad \vec{R}_{\text{NL}} = \begin{pmatrix} 0 \\ R_{\text{NL}} \end{pmatrix}, \quad \vec{R}_W = \begin{pmatrix} 0 \\ R_W \end{pmatrix}, \quad (3.22)$$

and

$$\vec{\text{Mod}} = \sum_k \frac{\dot{\lambda}_k}{\lambda_k} \vec{\theta}_k \vec{Z}_{\ell_k}^{\Lambda} + \sum_k \frac{\dot{\mathbf{y}}_k}{\lambda_k} \cdot \vec{\theta}_k \vec{Z}_{\ell_k}^{\nabla}. \quad (3.23)$$

**Step 3.** Now, we derive the equations of  $\lambda_k$  and  $\mathbf{y}_k$  from the orthogonality (3.5). First,

$$\frac{d}{dt} (\varepsilon, \theta_1(\Lambda W_{\ell_1}))_{\dot{H}_{\ell_1}^1} = (\varepsilon_t, \theta_1(\Lambda W_{\ell_1}))_{\dot{H}_{\ell_1}^1} + (\varepsilon, \partial_t (\theta_1(\Lambda W_{\ell_1})))_{\dot{H}_{\ell_1}^1} = 0$$

Thus, using (3.17),

$$\begin{aligned} 0 &= (\eta, \theta_1(\Lambda W_{\ell_1}))_{\dot{H}_{\ell_1}^1} - \left( \varepsilon, \frac{\ell_1}{\lambda_1} \cdot \theta_1(\nabla(\Lambda W_{\ell_1})) \right)_{\dot{H}_{\ell_1}^1} \\ &\quad + \frac{\dot{\lambda}_1}{\lambda_1} \left( (\theta_1(\Lambda W_{\ell_1}), \theta_1(\Lambda W_{\ell_1}))_{\dot{H}_{\ell_1}^1} - (\varepsilon, (\theta_1(\Lambda^2 W_{\ell_1})))_{\dot{H}_{\ell_1}^1} \right) \\ &\quad + \left( \frac{\dot{\mathbf{y}}_1}{\lambda_1} \cdot \theta_1(\nabla W_{\ell_1}), \theta_1(\Lambda W_{\ell_1}) \right)_{\dot{H}_{\ell_1}^1} - \left( \varepsilon, \frac{\dot{\mathbf{y}}_1}{\lambda_1} \cdot \theta_1(\nabla \Lambda W_{\ell_1}) \right)_{\dot{H}_{\ell_1}^1} \\ &\quad + \sum_{k=2}^K \left( \frac{\dot{\lambda}_k}{\lambda_k} (\theta_k(\Lambda W_{\ell_k}), \theta_1(\Lambda W_{\ell_1}))_{\dot{H}_{\ell_1}^1} + \left( \frac{\dot{\mathbf{y}}_k}{\lambda_k} \cdot \theta_1(\nabla W_{\ell_k}), \theta_1(\Lambda W_{\ell_1}) \right)_{\dot{H}_{\ell_1}^1} \right). \end{aligned} \quad (3.24)$$

By the decay properties of  $W_{\ell}$  and integration by parts, we note that

$$\left| (\eta, \theta_1(\Lambda W_{\ell_1}))_{\dot{H}_{\ell_1}^1} \right| + \left| \left( \varepsilon, \frac{\ell_1}{\lambda_1} \cdot \theta_1(\nabla(\Lambda W_{\ell_1})) \right)_{\dot{H}_{\ell_1}^1} \right| \lesssim \|\vec{\varepsilon}\|_E. \quad (3.25)$$

Next, by (2.1),

$$(\theta_1(\Lambda W_{\ell_1}), \theta_1(\Lambda W_{\ell_1}))_{\dot{H}_{\ell_1}^1} - (\varepsilon, (\theta_1(\Lambda^2 W_{\ell_1})))_{\dot{H}_{\ell_1}^1} = (1 - |\ell_1|^2)^{\frac{1}{2}} \|\Lambda W\|_{\dot{H}^1}^2 + O(\|\varepsilon\|_E),$$

and by parity,

$$\left( \frac{\dot{\mathbf{y}}_1}{\lambda_1} \cdot \theta_1(\nabla W_{\ell_1}), \theta_1(\Lambda W_{\ell_1}) \right)_{\dot{H}_{\ell_1}^1} = 0, \quad \left( \varepsilon, \frac{\dot{\mathbf{y}}_1}{\lambda_1} \cdot \theta_1(\nabla \Lambda W_{\ell_1}) \right)_{\dot{H}_{\ell_1}^1} = O(\|\dot{\mathbf{y}}_1\| \|\varepsilon\|_E).$$

Concerning the last terms, we claim, for  $k \in \{2, \dots, K\}$ ,

$$\left| \frac{\dot{\lambda}_k}{\lambda_k} (\theta_k(\Lambda W_{\ell_k}), \theta_1(\Lambda W_{\ell_1}))_{\dot{H}_{\ell_1}^1} \right| + \left| \left( \frac{\dot{\mathbf{y}}_k}{\lambda_k} \cdot \theta_1(\nabla W_{\ell_1}), \theta_1(\Lambda W_{\ell_1}) \right)_{\dot{H}_{\ell_1}^1} \right| \lesssim \frac{1}{t^3} \left( \left| \frac{\dot{\lambda}_k}{\lambda_k} \right| + \left| \frac{\dot{\mathbf{y}}_k}{\lambda_k} \right| \right). \quad (3.26)$$

Indeed, estimate (3.26) is a direct consequence of the following technical result.

**Claim 2.** *Let  $0 < r_2 \leq r_1$  be such that  $r_1 + r_2 > \frac{5}{3}$ . For  $t$  large, the following hold.*

$$- \text{If } r_1 > \frac{5}{3} \text{ then } \int |W_1|^{r_1} |W_2|^{r_2} \lesssim t^{-3r_2}, \quad (3.27)$$

$$- \text{If } r_1 \leq \frac{5}{3} \text{ then } \int |W_1|^{r_1} |W_2|^{r_2} \lesssim t^{5-3(r_1+r_2)}. \quad (3.28)$$

*Proof of Claim 2.* Estimates written in this proof are for  $t$  large enough, and all constants may depend on  $\ell_k$ . For convenience, we denote

$$\rho_k = x - \ell_k t - \mathbf{y}_k(t), \quad \Omega_k(t) = \{x \text{ such that } |\rho_k| < |\ell_1 - \ell_2|t/10\}.$$

Note that, for  $t$  large,

$$\begin{aligned} \text{for } x \in \Omega_2, \quad |W_1(x)| &\lesssim \frac{1}{\langle \rho_1 \rangle^3} \lesssim \frac{1}{(\langle \rho_2 \rangle + t)^3}, \\ \text{for } x \in \Omega_2^C, \quad |W_2(x)| &\lesssim \frac{1}{(\langle \rho_2 \rangle + t)^3} \lesssim \frac{1}{t^3}, \\ \text{for } x \in \Omega_1^C, \quad |W_1(x)| &\lesssim \frac{1}{(\langle \rho_1 \rangle + t)^3} \lesssim \frac{1}{t^3}, \end{aligned}$$

Case  $r_1 > \frac{5}{3}$ ,  $r_2 > \frac{5}{3}$ . Then,

$$\int_{\Omega_2} |W_1|^{r_1} |W_2|^{r_2} \lesssim t^{-3r_1} \int |W_2|^{r_2} \lesssim t^{-3r_1}, \quad \int_{\Omega_2^C} |W_1|^{r_1} |W_2|^{r_2} \lesssim t^{-3r_2} \int |W_1|^{r_1} \lesssim t^{-3r_2}.$$

Case  $r_1 > \frac{5}{3}$ ,  $0 < r_2 \leq \frac{5}{3}$ . In this case,

$$\begin{aligned} \int_{\Omega_2} |W_1|^{r_1} |W_2|^{r_2} &\lesssim \int \frac{1}{(\langle \rho_2 \rangle + t)^{3r_1}} \frac{1}{\langle \rho_2 \rangle^{3r_2}} dx \\ &\lesssim \int \frac{1}{(\langle x \rangle + t)^{3r_1}} \frac{dx}{\langle x \rangle^{3r_2}} \lesssim t^{-3(r_1+r_2)+5} \lesssim t^{-3r_2}, \end{aligned}$$

and

$$\int_{\Omega_2^C} |W_1|^{r_1} |W_2|^{r_2} \lesssim t^{-3r_2} \int |W_1|^{r_1} \lesssim t^{-3r_2}.$$

Case  $0 < r_1 \leq \frac{5}{3}$ ,  $0 < r_2 \leq \frac{5}{3}$ ,  $r_1 + r_2 > \frac{5}{3}$ . First, as before,

$$\int_{\Omega_2} |W_1|^{r_1} |W_2|^{r_2} \lesssim t^{-3(r_1+r_2)+5}, \quad \int_{\Omega_1} |W_1|^{r_1} |W_2|^{r_2} \lesssim t^{-3(r_1+r_2)+5}.$$

Next, by Holder inequality,

$$\begin{aligned} \int_{(\Omega_1 \cup \Omega_2)^C} |W_1|^{r_1} |W_2|^{r_2} &\lesssim \left( \int_{\Omega_1^C} \frac{1}{(\langle \rho_1 \rangle + t)^{3(r_1+r_2)}} \right)^{\frac{r_1}{r_1+r_2}} \left( \int_{\Omega_2^C} \frac{1}{(\langle \rho_2 \rangle + t)^{3(r_1+r_2)}} \right)^{\frac{r_2}{r_1+r_2}} \\ &\lesssim t^{-3(r_1+r_2)+5}. \end{aligned}$$

The claim is proved  $\square$

In conclusion of the previous estimates, the orthogonality condition  $(\varepsilon, \theta_1(\Lambda W_{\ell_1}))_{\dot{H}_{\ell_1}^1} = 0$ , gives the following

$$|\dot{\lambda}_1| \lesssim \|\vec{\varepsilon}\|_E + |\dot{\mathbf{y}}_1| \|\vec{\varepsilon}\|_E + \frac{1}{t^3} \sum_{k=1}^K \left( |\dot{\lambda}_k| + |\dot{\mathbf{y}}_k| \right). \quad (3.29)$$

Using the other orthogonality conditions, we obtain similarly, for  $k = 1, \dots, 5$ ,

$$|\dot{\lambda}_k| \lesssim \|\vec{\varepsilon}\|_E + |\dot{\mathbf{y}}_k| \|\vec{\varepsilon}\|_E + \frac{1}{t^3} \sum_{k'=1}^K \left( |\dot{\lambda}_{k'}| + |\dot{\mathbf{y}}_{k'}| \right), \quad (3.30)$$

$$|\dot{\mathbf{y}}_k| \lesssim \|\vec{\varepsilon}\|_E + |\dot{\lambda}_k| \|\vec{\varepsilon}\|_E + \frac{1}{t^3} \sum_{k'=1}^K \left( |\dot{\lambda}_{k'}| + |\dot{\mathbf{y}}_{k'}| \right). \quad (3.31)$$

Combining these estimates, we find (3.11). Note that equation (3.24) and the corresponding formula for  $\dot{\lambda}_k$  and  $\dot{\mathbf{y}}_k$  for  $k \geq 1$ , where  $\vec{\varepsilon}$  is replaced by  $\vec{u} - \sum_k \vec{W}_k$  form a nondegenerate first order differential system, whose unique solution is  $(\lambda_k, \mathbf{y}_k)_k$ , which justifies the  $C^1$  regularity of the parameters.

**Step 4.** Unstable directions. Recall that the quantities  $z_k^\pm$  are defined through the  $L^2$  scalar product  $z_k^\pm(t) = \left( \vec{\varepsilon}(t), \vec{\theta}_k \vec{Z}_{\ell_k}^\pm \right)_{L^2}$ . Recall also that  $\vec{Z}_{\ell_k}^\pm \in \mathcal{S}$ . By (3.21), we have

$$\begin{aligned} \frac{d}{dt} z_1^\pm &= \frac{d}{dt} \left( \vec{\varepsilon}, \vec{\theta}_1 \vec{Z}_{\ell_1}^\pm \right)_{L^2} = \left( \vec{\varepsilon}_t, \vec{\theta}_1 \vec{Z}_{\ell_1}^\pm \right)_{L^2} + \left( \vec{\varepsilon}, \partial_t \left( \vec{\theta}_1 \vec{Z}_{\ell_1}^\pm \right) \right)_{L^2} \\ &= \left( \vec{\mathcal{L}} \vec{\varepsilon}, \vec{\theta}_1 \vec{Z}_{\ell_1}^\pm \right)_{L^2} + \frac{\ell_1}{\lambda_1} \cdot \left( \vec{\varepsilon}, \vec{\theta}_1 \nabla \vec{Z}_{\ell_1}^\pm \right)_{L^2} \\ &+ \frac{\dot{\lambda}_1}{\lambda_1} \left( \left( \vec{\theta}_1 \vec{Z}_{\ell_1}^\Lambda, \vec{\theta}_1 \vec{Z}_{\ell_1}^\pm \right)_{L^2} - \left( \vec{\varepsilon}, \vec{\theta}_1 \vec{\Lambda} \vec{Z}_{\ell_1}^\pm \right)_{L^2} \right) + \frac{\dot{\mathbf{y}}_1}{\lambda_1} \cdot \left( \left( \vec{\theta}_1 \vec{Z}_{\ell_1}^\nabla, \vec{\theta}_1 \vec{Z}_{\ell_1}^\pm \right)_{L^2} - \left( \vec{\varepsilon}, \vec{\theta}_1 \nabla \vec{Z}_{\ell_1}^\pm \right)_{L^2} \right) \\ &+ \sum_{k=2}^K \left( \frac{\dot{\lambda}_k}{\lambda_k} \left( \vec{\theta}_k \vec{Z}_{\ell_k}^\Lambda, \vec{\theta}_1 \vec{Z}_{\ell_1}^\pm \right)_{L^2} + \frac{\dot{\mathbf{y}}_k}{\lambda_k} \cdot \left( \vec{\theta}_k \vec{Z}_{\ell_k}^\nabla, \vec{\theta}_1 \vec{Z}_{\ell_1}^\pm \right)_{L^2} \right) + \left( \vec{R}_{\text{NL}} + \vec{R}_W, \vec{\theta}_1 \vec{Z}_{\ell_1}^\pm \right)_{L^2}. \end{aligned}$$

First, by direct computations, using (2.40),

$$\begin{aligned}
& \left( \vec{\mathcal{L}}\vec{\varepsilon}, \vec{\theta}_1 \vec{Z}_{\ell_1}^\pm \right)_{L^2} - \frac{\ell_1}{\lambda_1} \cdot \left( \vec{\varepsilon}, \vec{\theta}_1 \nabla \vec{Z}_{\ell_1}^\pm \right)_{L^2} \\
&= \frac{1}{\lambda_1} \left( \vec{\varepsilon}, \vec{\theta}_1 \left( -H_{\ell_1} J \vec{Z}_{\ell_1}^\pm \right) \right)_{L^2} + \sum_{k \geq 2} \left( \varepsilon, f'(\theta_k W_{\ell_k})(\theta_1 Z_{\ell_{1,2}}^\pm) \right)_{L^2} \\
&= \pm \frac{\sqrt{\lambda_0}}{\lambda_1} (1 - |\ell_1|^2)^{\frac{1}{2}} z_1^\pm + \sum_{k \geq 2} \left( \varepsilon, f'(\theta_k W_{\ell_k})(\theta_1 Z_{\ell_{1,2}}^\pm) \right)_{L^2}.
\end{aligned}$$

Note that by the decay properties of  $\vec{Z}_{\ell_1}^\pm$  and Claim 2, for  $k \geq 2$ ,

$$\left| \left( \varepsilon, f'(\theta_k W_{\ell_k})(\theta_1 Z_{\ell_{1,2}}^\pm) \right)_{L^2} \right| \lesssim \frac{\|\varepsilon\|_{\dot{H}^1}}{t^4}. \quad (3.32)$$

By (2.39), we have

$$\left( \vec{\theta}_1 \vec{Z}_{\ell_1}^\Lambda, \vec{\theta}_1 \vec{Z}_{\ell_1}^\pm \right)_{L^2} = \left( \vec{Z}_{\ell_1}^\Lambda, \vec{Z}_{\ell_1}^\pm \right)_{L^2} = 0,$$

and thus, by (3.11),

$$\left| \frac{\dot{\lambda}_1}{\lambda_1} \left( \left( \vec{\theta}_1 \vec{Z}_{\ell_1}^\Lambda, \vec{\theta}_1 \vec{Z}_{\ell_1}^\pm \right)_{L^2} - \left( \vec{\varepsilon}, \vec{\theta}_1 \vec{\Lambda} \vec{Z}_{\ell_1}^\pm \right)_{L^2} \right) \right| \lesssim |\dot{\lambda}_1| \|\vec{\varepsilon}\|_E \lesssim \|\vec{\varepsilon}\|_E^2. \quad (3.33)$$

Similarly,

$$\left| \frac{\dot{\gamma}_1}{\lambda_1} \cdot \left( \left( \vec{\theta}_1 \vec{Z}_{\ell_1}^\nabla, \vec{\theta}_1 \vec{Z}_{\ell_1}^\pm \right)_{L^2} + \left( \vec{\varepsilon}, \vec{\theta}_1 \nabla \vec{Z}_{\ell_1}^\pm \right)_{L^2} \right) \right| \lesssim \|\vec{\varepsilon}\|_E^2. \quad (3.34)$$

Next, by Claim 2, we have

$$\left| \left( \vec{\theta}_2 \vec{Z}_{\ell_2}^\Lambda, \vec{\theta}_1 \vec{Z}_{\ell_1}^\pm \right)_{L^2} \right| + \left| \left( \vec{\theta}_2 \vec{Z}_{\ell_2}^\nabla, \vec{\theta}_1 \vec{Z}_{\ell_1}^\pm \right)_{L^2} \right| \lesssim \frac{1}{t^3}.$$

Thus, by (3.11),

$$\left| \frac{\dot{\lambda}_2}{\lambda_2} \left( \vec{\theta}_2 \vec{Z}_{\ell_2}^\Lambda, \vec{\theta}_1 \vec{Z}_{\ell_1}^\pm \right)_{L^2} \right| + \left| \frac{\dot{\gamma}_2}{\lambda_2} \cdot \left( \vec{\theta}_2 \vec{Z}_{\ell_2}^\nabla, \vec{\theta}_1 \vec{Z}_{\ell_1}^\pm \right)_{L^2} \right| \lesssim \frac{\|\vec{\varepsilon}\|_E}{t^3}. \quad (3.35)$$

Finally, we claim

$$\left| \left( \vec{R}_W, \vec{\theta}_1 \vec{Z}_{\ell_1}^\pm \right)_E \right| + \left| \left( \vec{R}_{\text{NL}}, \vec{\theta}_1 \vec{Z}_{\ell_1}^\pm \right)_E \right| \lesssim \frac{1}{t^3} + \frac{\|\varepsilon\|_{\dot{H}^1}}{t} + \|\varepsilon\|_{\dot{H}^1}^2. \quad (3.36)$$

Proof of (3.36). Note the following estimate, for any  $p > 1$ ,

$$|R_W| = \left| f \left( \sum_k W_k \right) - \sum_k f(W_k) \right| \lesssim \sum_{k \neq k'} |W_k|^{\frac{4}{3}} |W_{k'}|. \quad (3.37)$$

Thus, using Claim 2,

$$|(R_W, \theta_1(-\ell_1 \partial_{x_1} \Lambda W_{\ell_1}))_{L^2}| \lesssim \int \left( \sum_{k \neq k'} |W_k|^{\frac{4}{3}} |W_{k'}| \right) |W_1|^{\frac{4}{3}} \lesssim \frac{1}{t^3}. \quad (3.38)$$

Next, we decompose  $R_{\text{NL}} = R_{\varepsilon,1} + R_{\varepsilon,2}$ , where

$$\begin{aligned} R_{\varepsilon,1} &= \left( f' \left( \sum_k W_k \right) - \sum_k f'(W_k) \right) \varepsilon, \\ R_{\varepsilon,2} &= f \left( \sum_k W_k + \varepsilon \right) - f \left( \sum_k W_k \right) - f' \left( \sum_k W_k \right) \varepsilon. \end{aligned}$$

First,

$$|R_{\varepsilon,1}| \leq \left( \sum_{k' \neq k} |W_{k'}|^{\frac{1}{3}} |W_k| \right) |\varepsilon|.$$

Thus, using Claim 2 and (2.3)

$$|(R_{\varepsilon,1}, \theta_1(-\ell_1 \partial_{x_1} \Lambda W_{\ell_1}))_{L^2}| \lesssim \int \left( \sum_{k' \neq k} |W_{k'}|^{\frac{1}{3}} |W_k| \right) |W_1|^{\frac{4}{3}} |\varepsilon| \quad (3.39)$$

$$\lesssim \left( \int |\varepsilon|^2 |W_1|^{\frac{2}{3}} \right)^{\frac{1}{2}} \left( \int W_1^2 \left( \sum_{k' \neq k} |W_{k'}|^{\frac{2}{3}} |W_k|^2 \right) \right)^{\frac{1}{2}} \lesssim \frac{1}{t} \|\varepsilon\|_{\dot{H}^1}. \quad (3.40)$$

Finally, we have  $|R_{\varepsilon,2}| \lesssim \left( \sum_k |W_k|^{\frac{1}{3}} \right) |\varepsilon|^2 + |\varepsilon|^{\frac{7}{3}}$ , and thus, by (2.3) and (2.4),

$$|(R_{\varepsilon,2}, \theta_1(-\ell_1 \partial_{x_1} \Lambda W_{\ell_1}))_{L^2}| \lesssim \int \left( \left( \sum_k |W_k|^{\frac{1}{3}} \right) |\varepsilon|^2 + |\varepsilon|^{\frac{7}{3}} \right) |W_1|^{\frac{4}{3}} \lesssim \|\varepsilon\|_{\dot{H}^1}^2 + \|\varepsilon\|_{\dot{H}^1}^{\frac{7}{3}}.$$

The proof of (3.36) is complete.

Extending this computation to  $z_k^\pm$  for any  $k$ , we obtain in conclusion

$$\left| \frac{d}{dt} z_k^\pm(t) \mp \frac{\sqrt{\lambda_0}}{\lambda_k(t)} (1 - |\ell_k|^2)^{\frac{1}{2}} z_k^\pm(t) \right| \lesssim \|\tilde{\varepsilon}(t)\|_E^2 + \frac{\|\tilde{\varepsilon}(t)\|_E}{t} + \frac{1}{t^3}. \quad (3.41)$$

The proof of Lemma 3.1 is complete.  $\square$

#### 4. PROOF OF THEOREM 1 CASE (B)

In this section, we prove the existence of a solution  $u(t)$  of (1.1) satisfying (1.4)–(1.5) in case (B) of Theorem 1. We argue by compactness and obtain  $u(t)$  as the limit of suitable approximate multi-solitons  $u_n(t)$ .

Let  $K \geq 1$  and for all  $k \in \{1, \dots, K\}$ , let  $\lambda_k^\infty > 0$ ,  $\mathbf{y}_k^\infty \in \mathbb{R}^5$  and  $\ell_k \in \mathbb{R}^5$ . Let  $S_n \rightarrow +\infty$ . For  $\zeta_{k,n}^\pm \in \mathbb{R}$  small to be determined later (see statements of Proposition 4.1, Claim 3 and Lemma 4.2), we consider the solution  $u_n$  of

$$\begin{cases} \partial_t^2 u_n - \Delta u_n - |u_n|^{\frac{4}{3}} u_n = 0 \\ (u_n(S_n), \partial_t u_n(S_n)) = \sum_k \left[ (\vec{\theta}_k^\infty \vec{W}_{\ell_k})(S_n) + \zeta_{k,n}^+ (\vec{\theta}_k^\infty \vec{Z}_{\ell_k}^+)(S_n) + \zeta_{k,n}^- (\vec{\theta}_k^\infty \vec{Z}_{\ell_k}^-)(S_n) \right] \end{cases} \quad (4.1)$$

Note that since  $(u_n(S_n), \partial_t u_n(S_n)) \in Y^1 \times Y^0$ , the solution  $u_n$  is well-defined in  $Y^1 \times Y^0$  at least on a small interval of time around  $S_n$  (see section 2.1).

Now, we state the main uniform estimates on  $u_n$ .

**Proposition 4.1.** *Under the assumptions of Theorem 1, case (B), there exist  $n_0 > 0$  and  $T_0 > 0$  such that, for any  $n \geq n_0$ , there exist  $(\zeta_{k,n}^\pm)_{k \in \{1, \dots, K\}} \in \mathbb{R}^{2K}$ , with*

$$\sum_{k=1}^K |\zeta_{k,n}^\pm|^2 \lesssim \frac{1}{S_n^5}, \quad (4.2)$$

and such that the solution  $\vec{u}_n = (u_n, \partial_t u_n)$  of (4.1) is well-defined in  $Y^1 \times Y^0$  on the time interval  $[T_0, S_n]$  and satisfies

$$\forall t \in [T_0, S_n], \quad \left\| \vec{u}_n(t) - \sum_{k=1}^K \vec{W}_k^\infty \right\|_{\dot{H}^1 \times L^2} \lesssim \frac{1}{t}, \quad \left\| \vec{u}_n(t) - \sum_{k=1}^K \vec{W}_k^\infty \right\|_{Y^1 \times Y^0} \lesssim \frac{1}{t^{\frac{1}{2}}}. \quad (4.3)$$

**4.1. Proof of Theorem 1 case (B), assuming Proposition 4.1.** In view of the uniform bounds obtained in (4.3) at  $t = T_0$ , up to the extraction of a subsequence,  $(u_n(T_0), \partial_t u_n(T_0))$  converges strongly in  $\dot{H}^1 \times L^2$  to some  $(u_0, u_1)$  as  $n \rightarrow +\infty$ . Consider the solution  $u(t)$  of (1.1) associated to the initial data  $(u_0, u_1)$  at  $t = T_0$ . Then, by the uniform bounds (4.3) and the continuous dependence of the solution of (1.1) with respect to its initial data in the energy space  $\dot{H}^1 \times L^2$  (see e.g. [14] and references therein), the solution  $u$  is well-defined in the energy space on  $[T_0, \infty)$  and satisfies

$$\left\| \vec{u}(t) - \sum_{k=1}^K W_k^\infty \right\|_{\dot{H}^1 \times L^2} \lesssim \frac{1}{t}. \quad (4.4)$$

This finishes the proof of Theorem 1 in case (B), assuming Proposition 4.1.

The rest of this section is devoted to the proof of Proposition 4.1.

**4.2. Bootstrap setting.** We denote by  $B_{\mathbb{R}^K}(\rho)$  (respectively,  $S_{\mathbb{R}^K}(\rho)$ ) the ball (respectively, the sphere) of  $\mathbb{R}^K$  of center 0 and of radius  $\rho > 0$ , for the usual norm  $|(\xi_k)_k| = \left( \sum_{k=1}^K \xi_k^2 \right)^{1/2}$ .

For  $t = S_n$  and for  $t < S_n$  as long as  $u(t)$  is well-defined in  $\dot{H}^1 \times L^2$  and satisfies (3.3), we decompose  $u_n(t)$  as in Lemma 3.1. In particular, we denote by  $(\varepsilon, \eta)$ ,  $(\lambda_k)_k$ ,  $(\mathbf{y}_k)_k$ ,  $(z_k^\pm)_k$  the parameters of the decomposition of  $u_n$ . We also set

$$\mathcal{W}_K = \sum_{k=1}^K W_k, \quad \widetilde{\mathcal{W}}_K = \sum_{k=1}^K |W_k|. \quad (4.5)$$

We start with a technical result similar to Lemma 3 in [4]. This claim will allow us to adjust the initial values of  $(z_k^\pm(S_n))_k$  from the choice of  $\zeta_{k,n}^\pm$  in (4.1).

**Claim 3** (Choosing the initial unstable modes). *There exist  $n_0 > 0$  and  $C > 0$  such that, for all  $n \geq n_0$ , for any  $(\xi_k)_{k \in \{1, \dots, K\}} \in \overline{B}_{\mathbb{R}^K}(S_n^{-5/2})$ , there exists a unique  $(\zeta_{k,n}^\pm)_{k \in \{1, \dots, K\}} \in B_{\mathbb{R}^K}(CS_n^{-5/2})$  such that the decomposition of  $u_n(S_n)$  satisfies*

$$z_k^-(S_n) = \xi_k, \quad z_k^+(S_n) = 0, \quad (4.6)$$

$$|\lambda_k(S_n) - \lambda_k^\infty| + |\mathbf{y}_k(S_n) - \mathbf{y}_k^\infty| + \|\vec{\varepsilon}(S_n)\|_E \lesssim S_n^{-5/2}, \quad (4.7)$$

$$\|\vec{\varepsilon}(S_n)\|_{Y^1 \times Y^0} \lesssim S_n^{-2}. \quad (4.8)$$

*Sketch of the proof of Claim 3.* The proof of existence of  $(\zeta_{k,n}^\pm)_k$  in Claim 3 is similar to Lemma 3 in [4] and we omit it. Estimates in (4.7) are consequences of (3.6), (4.8) follows from (3.16).  $\square$

From now on, for any  $(\xi_k)_k \in \overline{B}_{\mathbb{R}^K}(S_n^{-5/2})$ , we fix  $(\zeta_{k,n}^\pm)_k$  as given by Claim 3 and the corresponding solution  $u_n$  of (4.1).

The proof of Proposition 4.1 is based on the following bootstrap estimates: for  $C^* > 1$  to be chosen,

$$\left. \begin{aligned} \sum_{k=1}^K |\lambda_k(t) - \lambda_k^\infty| + |\mathbf{y}_k(t) - \mathbf{y}_k^\infty| &\leq \frac{(C^*)^2}{t}, \quad \sum_{k=1}^K |z_k^\pm(t)|^2 \leq \frac{1}{t^5} \\ \|\tilde{\varepsilon}(t)\|_E &\leq \frac{C^*}{t^2}, \quad \|\tilde{\varepsilon}(t)\|_{Y^1 \times Y^0} \leq \frac{(C^*)^2}{t^{\frac{1}{2}}} \end{aligned} \right\} \quad (4.9)$$

Set

$$T^* = T_n^*((\xi_k)_k) = \inf\{t \in [T_0, S_n] ; u_n \text{ satisfies (3.3) and (4.9) holds on } [t, S_n]\}. \quad (4.10)$$

Note that by Claim 3, estimate (4.9) is satisfied at  $t = S_n$ . Moreover, if (4.9) is satisfied on  $[\tau, S_n]$  for some  $\tau \leq S_n$  then by the well-posedness theory in  $Y^1 \times Y^0$  and continuity,  $u_n(t)$  is well-defined and satisfies the decomposition of Lemma 3.1 on  $[\tau', S_n]$ , for some  $\tau' < \tau$ . In particular, the definition of  $T^*$  makes sense and it will suffice to strictly improve (4.9) on  $[T^*, S_n]$  to prove  $T^* = T_0$  for some  $(\xi_k)_k$ . Note also that we will prove that  $T^* = S_n$  for  $(\xi_k)_k \in \overline{B}_{\mathbb{R}^K}(S_n^{-5/2})$  (see proof of Lemma 4.2).

In what follows, we will prove that there exists  $T_0$  large enough and at least one choice of  $(\xi_k)_k \in \overline{B}_{\mathbb{R}^K}(S_n^{-5/2})$  so that  $T^* = T_0$ , which is enough to finish the proof of Proposition 4.1. For this, we derive general estimates for any  $(\xi_k)_k \in \overline{B}_{\mathbb{R}^K}(S_n^{-5/2})$  (see Lemma 4.1) and use a topological argument (see Lemma 4.2) to control the instable directions, in order to strictly improve estimates in (4.9) and thus prove that they cannot be saturated on  $[T_0, S_n]$ .

**4.3. Energy functional.** One of the main points of the proof of Proposition 4.1 is to derive suitable estimates in the energy norm that will strictly improve the bound on  $\|\tilde{\varepsilon}(t)\|_E$  from (4.9); the other estimates then follow easily.

We claim the following proposition in case (B) of Theorem 1. This is the only place in the paper where we need the restriction of collinear speeds.

**Proposition 4.2.** *Under the assumptions of Theorem 1, case (B), there exist  $\mu > 0$  and a function  $\mathcal{H}_K(t)$  on  $[T^*, S_n]$ , which satisfies the following properties.*

(i) Bound.

$$|\mathcal{H}_K(t)| \leq \frac{\|\tilde{\varepsilon}\|_E^2}{\mu}. \quad (4.11)$$

(ii) Coercivity.

$$\mathcal{H}_K(t) \geq \mu \|\tilde{\varepsilon}\|_E^2 - \frac{t^{-5}}{\mu}. \quad (4.12)$$

(iii) Time variation.

$$-\frac{d}{dt} (t^2 \mathcal{H}_K)(t) \lesssim C^* t^{-3}. \quad (4.13)$$

*Proof of Proposition 4.2.* We consider the case where the  $K$  solitons are moving in the same direction. In particular, by rotation invariance, we assume

$$\forall k \in \{1, \dots, K\}, \quad \ell_k = \ell_k \mathbf{e}_1 \quad \text{where} \quad \ell_k \in (-1, 1). \quad (4.14)$$

Moreover, without loss of generality,

$$-1 < \ell_1 < \dots < \ell_K < 1.$$

Fix

$$\max_k (|\beta_k|) < \bar{\ell} < 1.$$

For

$$0 < \sigma < \frac{1}{10} \min(\ell_{k+1} - \ell_k)$$

small enough to be fixed, we set

$$\begin{aligned} \text{for } k = 1, \dots, K-1, \quad \ell_k^+ &= \ell_k + \sigma(\ell_{k+1} - \ell_k), \\ \text{for } k = 2, \dots, K, \quad \ell_k^- &= \ell_k - \sigma(\ell_k - \ell_{k-1}), \end{aligned}$$

and for  $t > 0$ ,

$$\Omega(t) = ((\ell_1^+ t, \ell_2^- t) \cup \dots \cup (\ell_{K-1}^+ t, \ell_K^- t)) \times \mathbb{R}^4, \quad \Omega^C(t) = \mathbb{R}^5 \setminus \Omega(t).$$

We consider the continuous function  $\chi_K(t, x) = \chi_K(t, x_1)$  defined as follows, for all  $t > 0$ ,

$$\begin{cases} \chi_K(t, x) = \ell_1 \text{ for } x_1 \in (-\infty, \ell_1^+ t], \\ \chi_K(t, x) = \ell_k \text{ for } x_1 \in [\ell_k^- t, \ell_k^+ t], \text{ for } k \in \{2, \dots, K-1\}, \\ \chi_K(t, x) = \ell_K \text{ for } x_1 \in [\ell_K^- t, +\infty), \\ \chi_K(t, x) = \frac{x_1}{(1-2\sigma)t} - \frac{\sigma}{1-2\sigma}(\ell_{k+1} + \ell_k) \text{ for } x_1 \in [\ell_k^+ t, \ell_{k+1}^- t], k \in \{1, \dots, K-1\}. \end{cases} \quad (4.15)$$

In particular,

$$\begin{cases} \partial_t \chi_K(t, x) = 0, \quad \nabla \chi_K(t, x) = 0, \quad \text{on } \Omega^C(t), \\ \partial_{x_1} \chi_K(t, x) = \frac{1}{(1-2\sigma)t} \quad \text{for } x \in \Omega(t), \\ \partial_t \chi_K(t, x) = -\frac{1}{t} \frac{x_1}{(1-2\sigma)t} \quad \text{for } x \in \Omega(t). \end{cases} \quad (4.16)$$

We define

$$\mathcal{H}_K(t) = \int \mathcal{E}_K(t, x) dx + 2 \int (\chi_K(t, x) \partial_{x_1} \varepsilon(t, x)) \eta(t, x) dx,$$

where

$$\mathcal{E}_K = |\nabla \varepsilon|^2 + |\eta|^2 - 2(F(\mathcal{W}_K + \varepsilon) - F(\mathcal{W}_K) - f(\mathcal{W}_K) \varepsilon). \quad (4.17)$$

Note that from (4.9) and (3.11), we have

$$\sum_k (|\dot{\lambda}_k| + |\dot{y}_k|) \lesssim \|\vec{\varepsilon}(t)\|_E \lesssim \frac{C^*}{t^2}.$$

In particular, from (3.9) and (3.10), for all  $p \in \mathbb{N}^5$  (here  $|p| = \sum_j p_j$ ),

$$|\partial_x^p \text{Mod}_\varepsilon(t)| \lesssim \frac{C^*}{t^2} \widetilde{\mathcal{W}}_K^{1+\frac{|p|}{3}}, \quad |\partial_x^p \text{Mod}_\eta(t)| \lesssim \frac{C^*}{t^2} \widetilde{\mathcal{W}}_K^{\frac{4}{3}+\frac{|p|}{3}}. \quad (4.18)$$

*Proof of (4.11).* Since

$$|F(\mathcal{W}_K + \varepsilon) - F(\mathcal{W}_K) - f(\mathcal{W}_K)\varepsilon| \lesssim |\varepsilon|^{\frac{10}{3}} + \widetilde{\mathcal{W}}_K^{\frac{4}{3}}|\varepsilon|^2,$$

the estimate (4.11) on  $\mathcal{H}_K$  follows from Hölder inequality, (2.4) and (4.9).

*Proof of (4.12).* Set

$$\mathcal{N}_\Omega(t) = \int_\Omega (|\nabla \varepsilon(t)|^2 + \eta^2(t) + 2(\chi_K(t)\partial_{x_1}\varepsilon(t))\eta(t))$$

and

$$\mathcal{N}_{\Omega^C}(t) = \int_{\Omega^C} (|\nabla \varepsilon(t)|^2 + \eta^2(t)).$$

Note that, since  $|\chi_K| < \bar{\ell}$ ,

$$\begin{aligned} \mathcal{N}_\Omega &= \bar{\ell} \int_\Omega \left| \frac{\chi_K}{\bar{\ell}} \partial_{x_1} \varepsilon + \eta \right|^2 + \int_\Omega |\bar{\nabla} \varepsilon|^2 + \int_\Omega \left( 1 - \frac{\chi_K^2}{\bar{\ell}} \right) (\partial_{x_1} \varepsilon)^2 + (1 - \bar{\ell}) \int \eta^2 \\ &\geq \bar{\ell} \int_\Omega \left| \frac{\chi_K}{\bar{\ell}} \partial_{x_1} \varepsilon + \eta \right|^2 + (1 - \bar{\ell}) \int_\Omega (|\nabla \varepsilon|^2 + \eta^2). \end{aligned} \quad (4.19)$$

To obtain (4.12), we will actually prove the following stronger property

$$\mathcal{H}_K(t) \geq \mathcal{N}_\Omega(t) + \mu \mathcal{N}_{\Omega^C}(t) - \frac{t^{-5}}{\mu} - \frac{t^{-4\alpha}}{\mu} \|\bar{\varepsilon}\|_E^2 - \frac{1}{\mu} \|\bar{\varepsilon}\|_E^3. \quad (4.20)$$

We decompose  $\mathcal{H}_K = \mathbf{f}_1 + \mathbf{f}_2 + \mathbf{f}_3$ , where

$$\begin{aligned} \mathbf{f}_1 &= \int |\nabla \varepsilon|^2 - \int \left( \sum_k f'(W_k) \right) \varepsilon^2 + \int \eta^2 + 2 \int (\chi_K \partial_{x_1} \varepsilon) \eta, \\ \mathbf{f}_2 &= -2 \int \left( F(\mathcal{W}_K + \varepsilon) - F(\mathcal{W}_K) - f(\mathcal{W}_K)\varepsilon - \frac{1}{2} f'(\mathcal{W}_K) \varepsilon^2 \right), \\ \mathbf{f}_3 &= \int \left( \sum_k f'(W_k) - f'(\mathcal{W}_K) \right) \varepsilon^2, \end{aligned}$$

We claim the following estimates

$$\mathbf{f}_1 \geq \mathcal{N}_\Omega + \mu \mathcal{N}_{\Omega^C} - \frac{t^{-5}}{\mu} - \frac{t^{-4\alpha}}{\mu} \|\bar{\varepsilon}\|_E^2, \quad (4.21)$$

$$|\mathbf{f}_2| + |\mathbf{f}_3| \lesssim \|\bar{\varepsilon}\|_E^3 + \frac{\|\bar{\varepsilon}\|_E^2}{t^2}. \quad (4.22)$$

Note that combining these estimates with (4.9) and taking  $T_0$  large enough (depending on  $C^*$ ), we obtain (4.20) and then (4.12) for some other  $\mu > 0$ .

*Proof of (4.21).* The main ingredient in the proof of (4.21) is Lemma 2.2. For  $\varphi$  defined in (2.8), set

$$\varphi_k(t, x) = \varphi \left( \frac{x - \ell_k \mathbf{e}_1 t - \mathbf{y}_k(t)}{\lambda_k(t)} \right).$$

We decompose  $\mathbf{f}_1$  as follows

$$\begin{aligned} \mathbf{f}_1 &= \mathcal{N}_\Omega + \sum_k \left( \int |\nabla \varepsilon|^2 \varphi_k^2 - \int f'(W_k) \varepsilon^2 + \int \eta^2 \varphi_k^2 + 2 \int (\chi_K \partial_{x_1} \varepsilon) \eta \varphi_k^2 \right) \\ &\quad + \int_{\Omega^C} (|\nabla \varepsilon|^2 + \eta^2 + 2\chi_K (\partial_{x_1} \varepsilon) \eta) \left( 1 - \sum_k \varphi_k^2 \right) \\ &\quad - \int_\Omega (|\nabla \varepsilon|^2 + \eta^2 + 2\chi_K (\partial_{x_1} \varepsilon) \eta) \left( \sum_k \varphi_k^2 \right) \\ &\quad + 2 \sum_k \int (\chi_K - \ell_k) (\partial_{x_1} \varepsilon) \eta \varphi_k^2 = \mathcal{N}_\Omega + \mathbf{f}_{1,1} + \mathbf{f}_{1,2} + \mathbf{f}_{1,3} + \mathbf{f}_{1,4}. \end{aligned}$$

By Lemma 2.2, the orthogonality conditions on  $\vec{\varepsilon}$  and a change of variable, we have

$$\mathbf{f}_{1,1} \geq \mu \int (|\nabla \varepsilon|^2 + \eta^2) \left( \sum_k \varphi_k^2 \right) - \frac{1}{\mu} \sum_k ((z_k^-)^2 + (z_k^+)^2).$$

Thus, using (4.9),

$$\mathbf{f}_{1,1} \geq \mu \int (|\nabla \varepsilon|^2 + \eta^2) \left( \sum_k \varphi_k^2 \right) - \frac{1}{\mu} \frac{1}{t^5} \geq \mu \int_{\Omega^C} (|\nabla \varepsilon|^2 + \eta^2) \left( \sum_k \varphi_k^2 \right) - \frac{1}{\mu} \frac{1}{t^5}.$$

Next, note that if  $x$  is such that  $\varphi_k(t, x) > \frac{1}{2}$ , then  $\varphi_{k'}(x) \lesssim t^{-4\alpha}$  for  $k' \neq k$ . Thus,

$$1 - \sum_k \varphi_k^2 \gtrsim -t^{-4\alpha}.$$

By direct computations (with the notation  $v_+ = \max(0, v)$ ),

$$\begin{aligned} \mathbf{f}_{1,2} &= \bar{\ell} \int_{\Omega^C} \left| \frac{\chi_K}{\bar{\ell}} \partial_{x_1} \varepsilon + \eta \right|^2 \left( 1 - \sum_k \varphi_k^2 \right) + \int_{\Omega^C} |\nabla \varepsilon|^2 \left( 1 - \sum_k \varphi_k^2 \right) \\ &\quad + \int_{\Omega^C} \left( 1 - \frac{\chi_K}{\bar{\ell}} \right) |\partial_{x_1} \varepsilon|^2 \left( 1 - \sum_k \varphi_k^2 \right) + (1 - \bar{\ell}) \int_{\Omega^C} \eta^2 \left( 1 - \sum_k \varphi_k^2 \right) \\ &\geq (1 - \bar{\ell}) \int_{\Omega^C} (|\nabla \varepsilon|^2 + \eta^2) \left( 1 - \sum_k \varphi_k^2 \right) - \frac{\|\vec{\varepsilon}\|_E^2}{t^{4\alpha}}. \end{aligned}$$

Also, we see easily that  $|\mathbf{f}_{1,3}| \lesssim t^{-4\alpha} \|\vec{\varepsilon}\|_E^2$ .

Finally, by the definition of  $\chi_K$  in (4.15), the decay property of  $\varphi$  and (4.9) (for a bound on  $\mathbf{y}_k$ ), we have

$$\|(\chi_K - \ell_k) \varphi_k\|_{L^\infty} \leq t^{-4\alpha}.$$

Thus,

$$|\mathbf{f}_{1,4}| \lesssim t^{-4\alpha} \|\vec{\varepsilon}\|_E^2.$$

Therefore, for some  $\mu > 0$ , and  $T_0$  large enough, we have

$$\mathbf{f}_{1,1} + \mathbf{f}_{1,2} + \mathbf{f}_{1,3} + \mathbf{f}_{1,4} \geq \mu \mathcal{N}_{\Omega^C} - \frac{1}{\mu} \frac{1}{t^5} - t^{-4\alpha} \|\vec{\varepsilon}\|_E^2.$$

Proof of (4.22). Using Hölder inequality, (2.4) and (4.9), we have

$$|\mathbf{f}_2| \lesssim \int |\varepsilon|^{\frac{10}{3}} + |\varepsilon|^3 \widetilde{\mathcal{W}}_K^{\frac{1}{3}} \lesssim \|\varepsilon\|_E^3.$$

Next, since by the decay property of  $W$ ,

$$\left| f'(\mathcal{W}_K) - \sum_k f'(W_k) \right| \lesssim \frac{\widetilde{\mathcal{W}}_K^{\frac{2}{3}}}{t^2},$$

using (2.3), we obtain

$$|\mathbf{f}_3| \lesssim \frac{1}{t^2} \int |\varepsilon|^2 \widetilde{\mathcal{W}}_K^{\frac{2}{3}} \lesssim \frac{\|\varepsilon\|_E^2}{t^2}.$$

*Proof of (4.13). Step 1.* First estimates. We decompose

$$\frac{d}{dt} \mathcal{H}_K = \int \partial_t \mathcal{E}_K + 2 \int \chi_K \partial_t ((\partial_{x_1} \varepsilon) \eta) + 2 \int (\partial_t \chi_K) (\partial_{x_1} \varepsilon) \eta = \mathbf{g}_1 + \mathbf{g}_2 + \mathbf{g}_3.$$

We claim the following estimates

$$\begin{aligned} \mathbf{g}_1 &= 2 \int \varepsilon (-\Delta \text{Mod}_\varepsilon - f'(\mathcal{W}_K) \text{Mod}_\varepsilon) + 2 \int \eta \text{Mod}_\eta \\ &\quad + 2 \int \left( \sum_k \ell_k \partial_{x_1} W_k \right) (f(\mathcal{W}_K + \varepsilon) - f(\mathcal{W}_K) - f'(\mathcal{W}_K) \varepsilon) + O\left(\frac{C^*}{t^5}\right), \end{aligned} \quad (4.23)$$

$$\begin{aligned} \mathbf{g}_2 &= -\frac{1}{(1-2\sigma)t} \int_\Omega (\eta^2 + (\partial_{x_1} \varepsilon)^2 - |\nabla \varepsilon|^2) \\ &\quad - 2 \int \chi_K (\partial_{x_1} \mathcal{W}_K) (f(\mathcal{W}_K + \varepsilon) - f(\mathcal{W}_K) - f'(\mathcal{W}_K) \varepsilon) \\ &\quad + 2 \int (\chi_K \partial_{x_1} \text{Mod}_\varepsilon) \eta - 2 \int \varepsilon \chi_K \partial_{x_1} \text{Mod}_\eta + O\left(\frac{C^*}{t^5}\right), \end{aligned} \quad (4.24)$$

$$\mathbf{g}_3 = -\frac{2}{(1-2\sigma)t} \int_\Omega \frac{x_1}{t} \partial_{x_1} \varepsilon \eta. \quad (4.25)$$

*Estimate on  $\mathbf{g}_1$ .* From direct computations and the definition of  $\text{Mod}_\varepsilon$  in (3.9), we have

$$\begin{aligned} \mathbf{g}_1 &= 2 \int (\nabla \varepsilon_t \cdot \nabla \varepsilon + \eta_t \eta - \varepsilon_t (f(\mathcal{W}_K + \varepsilon) - f(\mathcal{W}_K))) \\ &\quad + 2 \int \left( \sum_k \ell_k \partial_{x_1} W_k \right) (f(\mathcal{W}_K + \varepsilon) - f(\mathcal{W}_K) - f'(\mathcal{W}_K) \varepsilon) \\ &\quad + 2 \int \text{Mod}_\varepsilon (f(\mathcal{W}_K + \varepsilon) - f(\mathcal{W}_K) - f'(\mathcal{W}_K) \varepsilon) = \mathbf{g}_{1,1} + \mathbf{g}_{1,2} + \mathbf{g}_{1,3}. \end{aligned}$$

Using (3.7) and integration by parts,

$$\mathbf{g}_{1,1} = 2 \int \eta R_W + 2 \int (\nabla \varepsilon \cdot \nabla \text{Mod}_\varepsilon - (f(\mathcal{W}_K + \varepsilon) - f(\mathcal{W}_K)) \text{Mod}_\varepsilon + \eta \text{Mod}_\eta)$$

By Cauchy-Schwarz inequality, (3.37) and then (4.9),

$$\left| \int \eta R_W \right| \lesssim \|\eta\|_{L^2} \|R_W\|_{L^2} \lesssim \frac{\|\eta\|_{L^2}}{t^3} \lesssim \frac{C^*}{t^5}.$$

Thus,

$$\mathbf{g}_{1,1} + \mathbf{g}_{1,3} = 2 \int \varepsilon (-\Delta \text{Mod}_\varepsilon - f'(\mathcal{W}_K) \text{Mod}_\varepsilon) + 2 \int \eta \text{Mod}_\eta + O\left(\frac{C^*}{t^5}\right),$$

and (4.23) follows.

*Estimate on  $\mathbf{g}_2$ .*

$$\begin{aligned} \mathbf{g}_2 &= 2 \int (\chi_K \partial_{x_1} \varepsilon_t) \eta + 2 \int (\chi_K \partial_{x_1} \varepsilon) \eta_t \\ &= 2 \int (\chi_K \partial_{x_1} \eta) \eta + 2 \int (\chi_K \partial_{x_1} \varepsilon) (\Delta \varepsilon + (f(\mathcal{W}_K + \varepsilon) - f(\mathcal{W}_K)) + R_W) \\ &\quad + 2 \int (\chi_K \partial_{x_1} \text{Mod}_\varepsilon) \eta + 2 \int (\chi_K \partial_{x_1} \varepsilon) \text{Mod}_\eta. \end{aligned}$$

Note that by integration by parts and (4.16)

$$\begin{aligned} 2 \int (\chi_K \partial_{x_1} \eta) \eta + 2 \int (\chi_K \partial_{x_1} \varepsilon) \Delta \varepsilon &= - \int \partial_{x_1} \chi_K (\eta^2 + (\partial_{x_1} \varepsilon)^2 - |\overline{\nabla} \varepsilon|^2) \\ &= - \frac{1}{(1-2\sigma)t} \int_{\Omega} (\eta^2 + (\partial_{x_1} \varepsilon)^2 - |\overline{\nabla} \varepsilon|^2). \end{aligned}$$

Next, we observe

$$\begin{aligned} \int (\chi_K \partial_{x_1} \varepsilon) (f(\mathcal{W}_K + \varepsilon) - f(\mathcal{W}_K) \varepsilon) &= \int \chi_K \partial_{x_1} (F(\mathcal{W}_K + \varepsilon) - F(\mathcal{W}_K) - f(\mathcal{W}_K) \varepsilon) \\ &\quad - \int \chi_K (\partial_{x_1} \mathcal{W}_K) (f(\mathcal{W}_K + \varepsilon) - f(\mathcal{W}_K) - f'(\mathcal{W}_K) \varepsilon). \end{aligned}$$

Moreover, integrating by parts and using (4.16),

$$\begin{aligned} &- \int \chi_K \partial_{x_1} (F(\mathcal{W}_K + \varepsilon) - F(\mathcal{W}_K) - f(\mathcal{W}_K) \varepsilon) \\ &= \frac{1}{(1-2\sigma)t} \int_{\Omega} (F(\mathcal{W}_K + \varepsilon) - F(\mathcal{W}_K) - f(\mathcal{W}_K) \varepsilon). \end{aligned}$$

Thus, by (4.9) and the decay of  $W$ ,

$$\left| \int \chi_K \partial_{x_1} (F(\mathcal{W}_K + \varepsilon) - F(\mathcal{W}_K) - f(\mathcal{W}_K) \varepsilon) \right| \lesssim \frac{1}{t} \int_{\Omega} \left( |\varepsilon|^{\frac{10}{3}} + \mathcal{W}_K^{\frac{4}{3}} |\varepsilon|^2 \right) \lesssim \frac{1}{t^5}.$$

Last, integrating by parts,

$$\begin{aligned} 2 \int (\chi_K \partial_{x_1} \varepsilon) \text{Mod}_\eta &= -2 \int (\chi_K \varepsilon) \partial_{x_1} \text{Mod}_\eta - 2 \int (\partial_{x_1} \chi_K) \varepsilon \text{Mod}_\eta \\ &= -2 \int (\chi_K \varepsilon) \partial_{x_1} \text{Mod}_\eta + O\left(\frac{1}{t^5}\right). \end{aligned}$$

Indeed, by (4.9), (4.16), (4.18) and (2.3),

$$\left| \int (\partial_{x_1} \chi_K) \varepsilon \text{Mod}_\eta \right| \lesssim \frac{C^*}{t^3} \int_\Omega |\varepsilon| \widetilde{\mathcal{W}}_K^{\frac{4}{3}} \lesssim \frac{C^*}{t^3} \left( \int |\varepsilon|^2 \widetilde{\mathcal{W}}_K^{\frac{2}{3}} \right)^{\frac{1}{2}} \left( \int_\Omega \widetilde{\mathcal{W}}_K^2 \right)^{\frac{1}{2}} \lesssim \frac{(C^*)^2}{t^{\frac{11}{2}}} \lesssim \frac{1}{t^5}.$$

*Estimate on  $\mathbf{g}_3$ .* (4.25) is a consequence of (4.16).

**Step 2.** Using cancellations and conclusion. In conclusion of estimates (4.23)–(4.25),

$$\frac{d}{dt} \mathcal{H}_K = \mathbf{h}_1 + \mathbf{h}_2 + \mathbf{h}_3 + \mathbf{h}_4 + O\left(\frac{C^*}{t^5}\right),$$

where

$$\begin{aligned} \mathbf{h}_1 &= -\frac{1}{(1-2\sigma)t} \int_\Omega \left( \eta^2 + (\partial_{x_1} \varepsilon)^2 + 2\frac{x_1}{t} (\partial_{x_1} \varepsilon) \eta - |\nabla \varepsilon|^2 \right), \\ \mathbf{h}_2 &= 2 \int \left( \sum_k (\ell_k - \chi_K) \partial_{x_1} W_k \right) (f(\mathcal{W}_K + \varepsilon) - f(\mathcal{W}_K) - f'(\mathcal{W}_K) \varepsilon), \\ \mathbf{h}_3 &= 2 \int \eta (\text{Mod}_\eta + \chi_K \partial_{x_1} \text{Mod}_\varepsilon), \\ \mathbf{h}_4 &= 2 \int \varepsilon (-\Delta \text{Mod}_\varepsilon - \chi_K \partial_{x_1} \text{Mod}_\eta - f'(\mathcal{W}_K) \text{Mod}_\varepsilon). \end{aligned}$$

First, by (4.19) and the definition of  $\chi_K$  in (4.15),

$$\begin{aligned} -((1-2\sigma)t) \mathbf{h}_1 &\leq \bar{\ell} \int_\Omega \left| \frac{\chi_K}{\bar{\ell}} \partial_{x_1} \varepsilon + \eta \right|^2 + \int_\Omega \left( 1 - \frac{\chi_K^2}{\bar{\ell}} \right) (\partial_{x_1} \varepsilon)^2 + (1 - \bar{\ell}) \int \eta^2 \\ &\quad + 2 \int_\Omega \left( \frac{x_1}{t} - \chi_K \right) \partial_{x_1} \varepsilon \eta \leq \mathcal{N}_\Omega + C\sigma \int (|\partial_{x_1} \varepsilon|^2 + \eta^2) \leq (1 + C\sigma) \mathcal{N}_\Omega. \end{aligned}$$

Second, we observe that by the definition of  $\chi_K$  in (4.16) and the decay of  $\partial_{x_1} W$  and  $W$ ,

$$|(\ell_k - \chi_K) \partial_{x_1} W_k| \lesssim |\ell_k - \chi_K| |W_k|^{4/3} \lesssim \frac{1}{t^2} |W_k|^{2/3}.$$

Thus, by (2.3) and (2.4),

$$|\mathbf{h}_2| \lesssim \frac{1}{t^2} \int \left( |\varepsilon|^{\frac{10}{3}} + |\varepsilon|^2 \widetilde{\mathcal{W}}_K^{\frac{2}{3}} \right) \lesssim \frac{(C^*)^2}{t^6} \lesssim \frac{1}{t^5}.$$

Denote

$$M_k = \frac{\dot{\lambda}_k}{\lambda_k} \Lambda W_k + \dot{\mathbf{y}}_k \cdot \nabla W_k \quad \text{so that} \quad \text{Mod}_\varepsilon = \sum_k M_k, \quad \text{Mod}_\eta = -\sum_k \ell_k \partial_{x_1} M_k$$

(see the definition of  $\text{Mod}_\varepsilon$  and  $\text{Mod}_\eta$  in (3.9)–(3.10)). Using (4.18), the definition of  $\chi_K$  (see (4.16)) and the decay of  $W$ ,

$$|(\ell_k - \chi_K) \partial_{x_1} M_k| \lesssim \frac{C^*}{t^2} \frac{1}{t^{\frac{13}{10}}} |W_k|^{\frac{9}{10}}. \quad (4.26)$$

In particular,

$$|\text{Mod}_\eta + \chi_K \partial_{x_1} \text{Mod}_\varepsilon| \lesssim \frac{C^*}{t^{\frac{33}{10}}} \widetilde{\mathcal{W}}_K^{\frac{9}{10}},$$

and thus, since  $\widetilde{\mathcal{W}}_K^{\frac{9}{10}}$  is bounded in  $L^2$ ,

$$|\mathbf{h}_3| = \left| \int \eta (\text{Mod}_\eta + \chi_K \partial_{x_1} \text{Mod}_\varepsilon) \right| \lesssim \frac{C^*}{t^{\frac{33}{10}}} \|\eta\|_{L^2} \lesssim \frac{(C^*)^2}{t^{\frac{53}{10}}} \lesssim \frac{1}{t^5}.$$

Finally, we see that by (2.21),  $-\Delta M_k + \ell_k^2 \partial_{x_1}^2 M_k - f'(W_k) M_k = 0$ . Thus, as before,

$$\begin{aligned} |-\Delta M_k + \ell_k \chi_K \partial_{x_1}^2 M_k - f'(\mathcal{W}_K) M_k| &\lesssim |(\chi_K - \ell_k) \partial_{x_1}^2 M_k| + |f'(\mathcal{W}_K) - f'(W_k)| |M_k| \\ &\lesssim \frac{C^*}{t^{\frac{33}{10}}} |W_k|^{\frac{37}{30}}. \end{aligned}$$

Therefore,

$$|-\Delta \text{Mod}_\varepsilon - \chi_K \partial_{x_1} \text{Mod}_\eta - f'(\mathcal{W}_K) \text{Mod}_\varepsilon| \lesssim \frac{C^*}{t^{\frac{33}{10}}} \widetilde{\mathcal{W}}_K^{\frac{37}{30}}.$$

It follows that (by (2.3)),

$$|\mathbf{h}_4| \lesssim \frac{C^*}{t^{\frac{33}{10}}} \left\| \varepsilon \widetilde{\mathcal{W}}_K^{1/3} \right\|_{L^2} \lesssim \frac{C^*}{t^{\frac{33}{10}}} \|\varepsilon\|_{\dot{H}^1} \lesssim \frac{(C^*)^2}{t^{\frac{53}{10}}} \lesssim \frac{1}{t^5}.$$

In conclusion, using (4.20), for  $\sigma$  small, and  $T_0$  large,

$$-\frac{d}{dt} \mathcal{H}_K \leq \frac{(1 + C\sigma)}{t} \mathcal{N}_\Omega + O\left(\frac{C^*}{t^5}\right) \leq \frac{2}{t} \mathcal{H}_K + O\left(\frac{C^*}{t^5}\right).$$

The proof of Proposition 4.2 is complete.  $\square$

**4.4. End of the proof of Proposition 4.1.** The following result, mainly based on Proposition 4.2, improves all the estimates in (4.9), except the ones on  $(z_k^-)_k$ .

**Lemma 4.1** (Closing estimates except  $(z_k^-)_k$ ). *For  $C^* > 0$  large enough, for all  $t \in [T^*, S_n]$ ,*

$$\left. \begin{aligned} |\lambda_k(t) - \lambda_k^\infty| + |\mathbf{y}_k(t) - \mathbf{y}_k^\infty| &\leq \frac{(C^*)^2}{2t}, \quad \sum_{k=1}^K |z_k^+(t)|^2 \leq \frac{1}{2t^5} \\ \|\tilde{\varepsilon}(t)\|_E &\leq \frac{C^*}{2t^2}, \quad \|\tilde{\varepsilon}(t)\|_{Y^1 \times Y^0} \leq \frac{(C^*)^2}{2t^{\frac{1}{2}}} \end{aligned} \right\} \quad (4.27)$$

The control of the directions  $(z_k^-)_k$ , related to the dynamical instability of  $W$ , requires a specific argument used in [4] in a similar context.

**Lemma 4.2** (Control of unstable directions). *There exist  $(\xi_{k,n})_k \in B_{\mathbb{R}^K}(S_n^{-5/2})$  such that, for  $C^* > 0$  large enough,  $T^*((\xi_{k,n})_k) = T_0$ . In particular, let  $(\zeta_n^\pm)$  be given by Claim 3 from such  $(\xi_{k,n})_k$ , then the solution  $u_n$  of (4.1) satisfies (4.3).*

Note that Lemma 4.2 completes the proof of Proposition 4.1.

*Proof of Lemma 4.1. Step 1.* We prove that for  $C^*$  large enough, for all  $t \in [T^*, S_n]$ ,

$$\|\tilde{\varepsilon}\|_{Y^1 \times Y^0} \leq \frac{(C^*)^2}{2t^{\frac{1}{2}}}. \quad (4.28)$$

The system (3.7) of equations of  $\varepsilon$  and  $\eta$  can be written under the form

$$\begin{cases} \varepsilon_t = \eta + \text{Mod}_\varepsilon \\ \eta_t = \Delta \varepsilon + R_\varepsilon + R_W + \text{Mod}_\eta, \end{cases}$$

where

$$|R_\varepsilon| \lesssim |\varepsilon|^{7/3} + |\varepsilon| \widetilde{\mathcal{W}}_K^{4/3}, \quad |\nabla R_\varepsilon| \lesssim |\nabla \varepsilon| \left( |\varepsilon|^{4/3} + \widetilde{\mathcal{W}}_K^{4/3} \right) + |\varepsilon| \widetilde{\mathcal{W}}_K^{4/3}.$$

In particular, by (2.6)

$$\|R_\varepsilon\|_{Y^0} \lesssim \|\varepsilon\|_{\dot{H}^1}^{\frac{1}{3}} \|\varepsilon\|_{Y^1}^2 + t^{\frac{1}{2}} \|\varepsilon\|_{\dot{H}^1} \lesssim C^* t^{-3/2}.$$

Moreover,

$$\|R_W\|_{Y_0} \lesssim t^{-5/2},$$

and by (4.18),

$$\|\text{Mod}_\varepsilon\|_{Y^1} + \|\text{Mod}_\eta\|_{Y^0} \lesssim C^* t^{-3/2}.$$

Using (4.8) and (2.5), we obtain

$$\begin{aligned} \|\tilde{\varepsilon}(t)\|_{Y^1 \times Y^0} &\lesssim \|\tilde{\varepsilon}(S_n)\|_{Y^1 \times Y^0} \\ &+ \int_t^{S_n} (\|R_\varepsilon(t')\|_{Y^0} + \|R_W(t')\|_{Y^0} + \|\text{Mod}_\varepsilon(t')\|_{Y^1} + \|\text{Mod}_\eta(t')\|_{Y^0}) dt' \lesssim \frac{C^*}{t^{\frac{1}{2}}}. \end{aligned}$$

In particular, taking  $C^*$  large enough, we obtain (4.28).

**Step 2.** Estimates on parameters. The estimates on  $|\lambda_k(t) - \lambda_k^\infty|$  and  $|\mathbf{y}_k(t) - \mathbf{y}_k^\infty|$  follow from integration of (3.11) using (4.9) and (4.7), and possibly taking a larger  $C^*$ .

Now, we prove the bound on  $z_k^+(t)$ . Let  $c_k = \frac{\sqrt{\lambda_0}}{\lambda_k^\infty} (1 - |\ell_k|^2)^{1/2} > 0$ . Then, from (3.13) and (4.9),

$$\frac{d}{dt} [e^{-c_k t} z_k^+] \lesssim e^{-c_k t} \frac{C^*}{t^3}.$$

Integrating on  $[t, S_n]$  and using (4.6), we obtain  $-z_k^+(t) \lesssim C^* t^{-3}$ . Doing the same for  $-e^{-c_k t} z_k^+$ , we obtain the conclusion for  $T_0$  large enough.

**Step 3.** Bound on the energy norm. Finally, to prove the estimate on  $\|\tilde{\varepsilon}(t)\|_E$ , we use Proposition 4.2. Recall from (4.7) and then (4.11) that

$$\mathcal{H}_K(S_n) \lesssim S_n^{-5}. \quad (4.29)$$

Integrating (4.13) on  $[t, S_n]$ , and using (4.29), we obtain, for all  $t \in [T^*, S_n]$ ,  $\mathcal{H}_K \lesssim C^* t^{-4}$ . Using (4.12), we conclude that  $\|\tilde{\varepsilon}\|_E^2 \lesssim C^* t^{-4}$ .  $\square$

*Proof of Lemma 4.2. Step 1.* Choice of  $(\zeta_k)$ . We follow the strategy of Lemma 6 in [4]. The proof is by contradiction, we assume that for any  $(\xi_k)_{k \in \{1, \dots, K\}} \in B_{\mathbb{R}^K}(S_n^{-5/2})$ ,  $T^*((\xi_k)_k)$  defined by (4.10) satisfies  $T^* \in (T_0, S_n)$ . In this case, by Lemma 4.1 and continuity, it holds necessarily

$$\sum_{k=1}^K |z_k^-(T^*)|^2 = \frac{1}{(T^*)^5}. \quad (4.30)$$

We claim the following transversality property at  $T^*$

$$\left. \frac{d}{dt} \left( t^5 \sum_{k=1}^K |z_k^-(t)|^2 \right) \right|_{t=T^*} < -\bar{c} < 0. \quad (4.31)$$

Let  $c_k = \frac{\sqrt{\lambda_0}}{\lambda_k^\infty} (1 - |\ell_k|^2)^{1/2} > 0$  and  $\bar{c} = \min_k c_k$ . From (3.13) and (4.9), for all  $t \in [T^*, S_n]$ ,

$$\begin{aligned} \frac{d}{dt} \left( t^5 (z_k^-)^2 \right) &= 2t^5 z_k^- \frac{d}{dt} z_k^- + 5t^4 (z_k^-)^2 \\ &\leq -2t^5 c_k (z_k^-)^2 + \frac{CC^*}{t^{\frac{1}{2}}} \leq -2\bar{c}t^5 (z_k^-)^2 + \frac{CC^*}{t^{\frac{1}{2}}}. \end{aligned}$$

Thus, from (4.30)

$$\left. \frac{d}{dt} \left( t^5 \sum_{k=1}^K (z_k^-)^2 \right) \right|_{t=T^*} \leq -2\bar{c} + \frac{CC^*}{(T^*)^{\frac{1}{2}}} < -\bar{c},$$

for  $T_0$  large enough (depending on  $C^*$ , but independent of  $n$ ).

As a consequence of (4.31), we observe that the map  $T^*$

$$(\xi_k)_{k \in \{1, \dots, K\}} \in \overline{B}_{\mathbb{R}^K}(S_n^{-5/2}) \mapsto T^*((\xi_k)_k)$$

is continuous. Indeed, if  $T^* < S_n$ , by (4.31), it is clear that for all  $\sigma > 0$  small enough, there exists  $\delta > 0$  so that for all  $t \in [T^* + \sigma, S_n]$ ,  $t^5 \sum_k (z_k^-(t))^2 < (1 - \delta)$ . In particular, for  $(\tilde{\xi}_k)_k \in B_{\mathbb{R}^K}(S_n^{-5/2})$  close enough to  $(\xi_k)_k$ , it follows that for all  $t \in [T^* + \sigma, S_n]$ ,  $t^5 \sum_k (\tilde{z}_k^-(t))^2 < (1 - \frac{1}{2}\delta)$ , and thus  $\tilde{T}^* < T^* + \sigma$ . By similar arguments, for  $(\tilde{\xi}_k)_k \in B_{\mathbb{R}^K}(S_n^{-5/2})$  close enough to  $(\xi_k)_k$ , we also have  $\tilde{T}^* > T^* - \sigma$ .

We define

$$\begin{aligned} \mathcal{M} : \overline{B}_{\mathbb{R}^K}(S_n^{-5/2}) &\rightarrow S_{\mathbb{R}^K}(S_n^{-5/2}) \\ (\xi_k)_k &\mapsto \left( \frac{T^*}{S_n} \right)^{5/2} (z_k^-(T^*))_k \end{aligned}$$

From what precedes,  $\mathcal{M}$  is continuous. Moreover, from (4.30) and (4.31),  $\mathcal{M}$  restricted to  $S_{\mathbb{R}^K}(S_n^{-5/2})$  is the identity (since in this case  $T^* = S_n$  and  $z_k^-(S_n) = \xi_k$  from (4.6)). The existence of such a map is contradictory with Brouwer's fixed point theorem.

**Step 2.** Conclusion. Proof of (4.3). These estimates follow directly from the estimates (4.9) on  $\varepsilon(t)$ ,  $\lambda_k(t)$ ,  $\mathbf{y}_k(t)$  and (3.14), (3.15).  $\square$

## 5. PROOF OF THEOREM 1 CASE (A) BY LORENTZ TRANSFORMATION

Let  $\lambda_1^\infty, \lambda_2^\infty > 0$ ,  $\mathbf{y}_1^\infty, \mathbf{y}_2^\infty \in \mathbb{R}^5$ ,  $\iota_1 = \pm 1$ ,  $\iota_2 = \pm 1$ . Let  $\ell_1, \ell_2 \in \mathbb{R}^5$  with  $\ell_1 \neq \ell_2$  and  $|\ell_k| < 1$  for  $k = 1, 2$ . We claim that there exists a solution  $u$  of (1.1) in the energy space, on a time interval  $[S_0, +\infty)$  such that (1.4) and (1.5) hold.

**Step 1.** Reduction of the problem by rotation. We change coordinates in  $\mathbb{R}^5$  so that by invariance of (1.1) by rotation, we reduce with loss of generality to the following case:

$$\ell_1 \cdot \mathbf{e}_1 = \ell_1, \quad \ell_2 \cdot \mathbf{e}_1 = \ell_2, \quad \ell_1 \cdot \mathbf{e}_2 = \ell_2 \cdot \mathbf{e}_2 := \beta, \quad \ell_1 \cdot \mathbf{e}_j = \ell_2 \cdot \mathbf{e}_j = 0, \quad \text{for } j = 3, 4, 5. \quad (5.1)$$

Indeed, it suffices to take as first vector of the new orthonormal basis  $\mathcal{B}'$  of  $\mathbb{R}^5$ , the vector  $\mathbf{e}'_1 = \frac{\ell_1 - \ell_2}{|\ell_1 - \ell_2|}$ , and as second vector  $\mathbf{e}'_2 = a\ell_1 + b\ell_2$ , where  $a$  and  $b$  are chosen so that  $\mathbf{e}'_1 \cdot \mathbf{e}'_2 = 0$  and  $|\mathbf{e}'_2| = 1$ . Then,  $\ell_1 \cdot \mathbf{e}'_2 = \ell_2 \cdot \mathbf{e}'_2$ . The basis  $\mathcal{B}'$  is then completed in any way.

Let  $\bar{x} = (x_3, x_4, x_5)$ .

Note that if  $\beta = 0$ , then  $\ell_k = \ell_k \mathbf{e}_1$  for  $k = 1, 2$  and then we are reduced to case (B) of Theorem 1 for  $K = 2$ . Now, we consider the general case  $0 < \beta < 1$ . Set

$$\tilde{\ell}_k = \frac{\ell_k}{\sqrt{1 - \beta^2}}, \quad |\tilde{\ell}_k| < 1, \quad k = 1, 2. \quad (5.2)$$

Also set ( $k = 1, 2$ )

$$\tilde{\mathbf{y}}_k^\infty \in \mathbb{R}^5 \quad \text{such that} \quad \begin{cases} \tilde{\mathbf{y}}_{k,1}^\infty = \mathbf{y}_{k,1}^\infty + \frac{\beta \ell_1}{1 - \beta^2} \mathbf{y}_{k,2}^\infty, \\ \tilde{\mathbf{y}}_{k,2}^\infty = \frac{\mathbf{y}_{k,2}^\infty}{\sqrt{1 - \beta^2}}, \\ \tilde{\mathbf{y}}_{k,j}^\infty = \mathbf{y}_{k,j}^\infty, \text{ for } j = 3, 4, 5. \end{cases} \quad (5.3)$$

For  $k = 1, 2$ , let

$$\tilde{W}_k^\infty(t, x) = \frac{\iota_k}{(\lambda_k^\infty)^{3/2}} W_{\tilde{\ell}_k} \left( \frac{x - \tilde{\ell}_k \mathbf{e}_1 t - \tilde{\mathbf{y}}_k^\infty}{\lambda_k^\infty} \right), \quad \vec{W}_k^\infty = (\tilde{W}_k^\infty, \partial_t \tilde{W}_k^\infty).$$

Let  $\tilde{u}(t)$  be the solution of (1.1) satisfying

$$\left\| \vec{u}(t) - \left[ \vec{W}_1^\infty(t) + \vec{W}_2^\infty(t) \right] \right\|_{\dot{H}^1 \times L^2} = 0 \quad (5.4)$$

given by Theorem 1, case (B). Define the Lorentz transform with parameter  $\beta \mathbf{e}_2$  of the solution  $\tilde{u}$ , i.e.

$$u(s, y) = \tilde{u} \left( \frac{s - \beta y_2}{\sqrt{1 - \beta^2}}, y_1, \frac{y_2 - \beta s}{\sqrt{1 - \beta^2}}, \bar{y} \right). \quad (5.5)$$

We claim that  $u(s, y)$  is a 2-soliton of (1.1) in the sense of Theorem 1 with parameters  $\lambda_k^\infty$ ,  $\mathbf{y}_k^\infty$  and speeds  $\ell_k \mathbf{e}_1 + \beta \mathbf{e}_2$ .

First, from the arguments of the proof of Lemma 6.1 in [9], since  $\tilde{u}(t, x)$  is well-defined on  $[T_0, +\infty)$  it is well-defined everywhere on the space-time domain  $\mathbb{R} \times \mathbb{R}^5$  except possibly in a half cone of the form  $t - t^- < -|x - x^-|$ , for some  $t^- \in \mathbb{R}$  and  $x^- \in \mathbb{R}^5$ . Thus, there exists  $S_0 \in \mathbb{R}$  such that  $u(s)$  defined by (5.5) makes sense on  $\mathbb{R}^5$  for all  $s > S_0$  (see also Lemma 5.1 below). Moreover, from the arguments of section 6 in [9] (see also section 2 of [14]),  $u$  is a finite energy solution of (1.1) on  $[S_0, +\infty)$ .

To prove the claim, we consider separately the regions “far from the solitons” and “close to the solitons”.

**Step 2.** Estimate far from the solitons. We claim that for all  $\delta > 0$ , there exists  $A_\delta > 0$  such that for all  $s \geq S_\delta$ ,

$$\|(u(s), \partial_t u(s))\|_{(\dot{H}^1 \times L^2)(|y - (\ell_k \mathbf{e}_1 + \beta \mathbf{e}_2)| > A_\delta)} \lesssim \delta. \quad (5.6)$$

Let  $\delta > 0$  and  $T_\delta > 0$  be such that

$$\sup_{t > T_\delta} \left\| \vec{u}(t) - \left[ \vec{W}_1^\infty(t) + \vec{W}_2^\infty(t) \right] \right\|_{\dot{H}^1 \times L^2} < \delta. \quad (5.7)$$

Moreover, let  $A_\delta > 1$  large enough so that for  $k = 1, 2$ ,

$$\sup_{t \in \mathbb{R}} \left\| \vec{W}_k^\infty(t) \right\|_{(\dot{H}^1 \times L^2)(|x - \tilde{\ell}_k \mathbf{e}_1 t| > A_\delta/2)} < \delta. \quad (5.8)$$

We recall the following result from section 2 of [14], Claim 6.7 and proof of Lemma 6.1 of [9] (and references therein for the small data Cauchy theory).

**Lemma 5.1** (Small scattering solutions and Lorentz transform [9]). *There exists  $\delta_0 > 0$  such that the following holds.*

- (i) *For all  $(w_0, w_1) \in \dot{H}^1 \times L^2$  such that  $\|(w_0, w_1)\|_{\dot{H}^1 \times L^2} < \delta_0$ , there exists a global scattering solution<sup>1</sup>  $(w(t), \partial_t w(t))$  of (1.1) with initial data  $(w_0, w_1)$ .*

*Moreover,  $\sup_{t \in \mathbb{R}} \|(w(t), \partial_t w(t))\|_{\dot{H}^1 \times L^2} \lesssim \delta_0$ .*

- (ii) *For  $(w, \partial_t w)$  as in (i) and  $\beta \in (-1, 1)$ , the function  $w_\beta(s, y)$  defined by*

$$w_\beta(s, y) = w\left(\frac{s - \beta y_2}{\sqrt{1 - \beta^2}}, y_1, \frac{y_2 - \beta s}{\sqrt{1 - \beta^2}}, \bar{y}\right) \quad (5.9)$$

*is a global scattering solution of (1.1). Moreover, for some constant  $C_\beta > 0$ ,*

$$\sup_{t \in \mathbb{R}} \|(w_\beta, \partial_t w_\beta)(t)\|_{\dot{H}^1 \times L^2} \leq C_\beta \|(w_0, w_1)\|_{\dot{H}^1 \times L^2}. \quad (5.10)$$

We defined a cutoff function  $\zeta \in C^\infty(\mathbb{R}^5)$  such that

$$\zeta(x) = 1 \text{ for } |x| > 1, \quad \zeta(x) = 0 \text{ for } |x| < \frac{1}{2}.$$

For  $t_0 > T_\delta$  to be chosen later, we also define

$$\zeta^{\text{ext}}(x) = \zeta\left(\frac{x - \tilde{\ell}_1 \mathbf{e}_1 t_0}{A_\delta}\right) \zeta\left(\frac{x - \tilde{\ell}_2 \mathbf{e}_1 t_0}{A_\delta}\right).$$

Define  $u^{\text{ext}}(t)$  the solution of (1.1) corresponding to the following initial data at  $t = t_0$ ,

$$u^{\text{ext}}(t_0, x) = \tilde{u}(t_0, x) \zeta^{\text{ext}}(x), \quad \partial_t u^{\text{ext}}(t_0, x) = (\partial_t \tilde{u}(t_0, x)) \zeta^{\text{ext}}(x).$$

By (5.7) and (5.8), choosing  $\delta > 0$  small enough (compared to  $\delta_0$ , given by Lemma 5.1), we have

$$\|(u^{\text{ext}}(t_0), \partial_t u^{\text{ext}}(t_0))\|_{\dot{H}^1 \times L^2} \leq \delta < \delta_0.$$

By Lemma 5.1,  $u^{\text{ext}}(t)$  is thus a global scattering solution of (1.1) on  $\mathbb{R} \times \mathbb{R}^5$ , and satisfies

$$\sup_{t \in \mathbb{R}} \|(u^{\text{ext}}(t), \partial_t u^{\text{ext}}(t))\|_{\dot{H}^1 \times L^2} \lesssim \delta.$$

Moreover, if we define  $u_\beta^{\text{ext}}(s, y)$  as the Lorentz transform with parameter  $\beta \mathbf{e}_2$  of  $u^{\text{ext}}$  (as in (5.9)), then  $u_\beta^{\text{ext}}$  is also a global scattering solution of (1.1) satisfying

$$\sup_{s \in \mathbb{R}} \|(u_\beta^{\text{ext}}(s), \partial_t u_\beta^{\text{ext}}(s))\|_{\dot{H}^1 \times L^2} \lesssim \delta. \quad (5.11)$$

Now, we deduce consequences of these observations on  $\tilde{u}$  and  $u$ . Indeed, since  $u^{\text{ext}}(t_0, x) = u(t_0, x)$ , and  $\partial_t u^{\text{ext}}(t_0, x) = \partial_t u(t_0, x)$ , for a.e.  $(t, x)$  such that  $|x - \tilde{\ell}_k \mathbf{e}_1 t_0| > A_\delta$  for  $k = 1, 2$ , it follows from finite speed of propagation that

$$u^{\text{ext}}(t, x) = \tilde{u}(t, x), \quad \partial_t u^{\text{ext}}(t, x) = \partial_t \tilde{u}(t, x) \quad \text{a.e. on } C_{A_\delta}(t_0),$$

where

$$C_{A_\delta}(t_0) = \{(t, x) \text{ such that } |x - \tilde{\ell}_1 \mathbf{e}_1 t_0| > A_\delta + |t - t_0| \text{ and } |x - \tilde{\ell}_2 \mathbf{e}_1 t_0| > A_\delta + |t - t_0|\}.$$

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<sup>1</sup>by global scattering solution, we mean a solution defined for all time  $t \in \mathbb{R}$  and behaving in the energy space as a free solution both as  $t \rightarrow +\infty$  and  $t \rightarrow -\infty$

Then, by the definitions of  $u$  and  $u_\beta^{\text{ext}}$ , for almost every  $(s, y)$  such that

$$\left( \frac{s - \beta y_2}{\sqrt{1 - \beta^2}}, y_1, \frac{y_2 - \beta s}{\sqrt{1 - \beta^2}}, \bar{y} \right) \in C_{A_\delta}(t_0),$$

we have

$$u^{\text{ext}}(s, y) = u(s, y), \quad \partial_s u^{\text{ext}}(s, y) = \partial_t u(s, y). \quad (5.12)$$

Now, let  $s_0 \geq S_\delta := \frac{T_\delta}{\sqrt{1 - \beta^2}}$  and choose  $t_0 = \sqrt{1 - \beta^2} s_0$ . By (5.11) and (5.12),

$$\begin{aligned} \|(u_\beta(s_0), \partial_t u_\beta(s_0))\|_{(\dot{H}^1 \times L^2)(\Omega_{A_\delta}(s_0))} &= \|(u_\beta^{\text{ext}}(s_0), \partial_t u_\beta^{\text{ext}}(s_0))\|_{(\dot{H}^1 \times L^2)(\Omega_{A_\delta}(s_0))} \\ &\leq \|(u_\beta^{\text{ext}}(s_0), \partial_t u_\beta^{\text{ext}}(s_0))\|_{\dot{H}^1 \times L^2} \lesssim \delta, \end{aligned} \quad (5.13)$$

where

$$\Omega_{A_\delta}(s_0) = \left\{ y \text{ such that } \left( \frac{s_0 - \beta y_2}{\sqrt{1 - \beta^2}}, y_1, \frac{y_2 - \beta s_0}{\sqrt{1 - \beta^2}}, \bar{y} \right) \in C_{A_\delta}(t_0) \right\}.$$

For  $C_\beta = \frac{2}{1 - |\beta|}$ , let

$$\Gamma_{A_\delta}(s_0) = \{ y \text{ such that } (|y_1 - \ell_k s_0|^2 + |y_2 - \beta s_0|^2 + |\bar{y}|^2) > C_\beta A_\delta \text{ for } k = 1 \text{ and } 2. \}$$

We claim that

$$\Omega_{A_\delta}(s_0) \supset \Gamma_{A_\delta}(s_0). \quad (5.14)$$

Indeed, for  $y \in \Gamma_{A_\delta}(s_0)$ , by the choice of  $t_0$ , for  $k = 1, 2$ ,

$$\begin{aligned} \left( |y_1 - \tilde{\ell}_k t_0|^2 + \frac{1}{1 - \beta^2} |y_2 - \beta s_0|^2 + |\bar{y}|^2 \right)^{1/2} &= \left( |y_1 - \ell_k s_0|^2 + \frac{1}{1 - \beta^2} |y_2 - \beta s_0|^2 + |\bar{y}|^2 \right)^{1/2} \\ &\geq (1 - |\beta|) (|y_1 - \ell_k s_0|^2 + |y_2 - \beta s_0|^2 + |\bar{y}|^2)^{1/2} + \frac{|\beta|}{\sqrt{1 - \beta^2}} |y_2 - \beta s_0| \\ &> A_\delta + \frac{|\beta|}{\sqrt{1 - \beta^2}} |y_2 - \beta s_0| = A_\delta + \left| \frac{s_0 - \beta y_2}{\sqrt{1 - \beta^2}} - t_0 \right|. \end{aligned}$$

Thus,  $y \in \Omega_{A_\delta}(s_0)$ .

Now, we observe that (5.14) and (5.13) prove (5.6).

**Step 3.** Estimate close to the solitons. First, we compute  $W_k^\infty(s, y)$ , the Lorentz transform with parameter  $\beta \mathbf{e}_2$  of  $\tilde{W}_k(t, x)$ . From the definition of  $\tilde{W}_k^\infty$ , (5.2) and (5.3),

$$\begin{aligned} W_k^\infty(s, y) &= \tilde{W}_k^\infty \left( \frac{s - \beta y_2}{\sqrt{1 - \beta^2}}, y_1, \frac{y_2 - \beta s}{\sqrt{1 - \beta^2}}, \bar{y} \right) \\ &= \frac{\iota_k}{(\lambda_k^\infty)^{3/2}} W \left( \frac{y_1 - \tilde{\ell}_k \left( \frac{s - \beta y_2}{\sqrt{1 - \beta^2}} \right) - \tilde{\mathbf{y}}_{k,1}^\infty}{\lambda_k^\infty \sqrt{1 - \tilde{\ell}_k^2}}, \frac{\frac{y_2 - \beta s}{\sqrt{1 - \beta^2}} - \tilde{\mathbf{y}}_{k,2}^\infty}{\lambda_k^\infty}, \frac{\bar{y} - \bar{\mathbf{y}}_k^\infty}{\lambda_k^\infty} \right) \\ &= \frac{\iota_k}{(\lambda_k^\infty)^{3/2}} W \left( \frac{(y_1 - \ell_k s - \mathbf{y}_{k,1}^\infty) + \frac{\beta \ell_k}{1 - \beta^2} (y_2 - \beta s - \mathbf{y}_{k,2}^\infty)}{\lambda_k^\infty \sqrt{1 - \frac{\ell_k^2}{1 - \beta^2}}}, \frac{y_2 - \beta s - \mathbf{y}_{k,2}^\infty}{\sqrt{1 - \beta^2} \lambda_k^\infty}, \frac{\bar{y} - \bar{\mathbf{y}}_k^\infty}{\lambda_k^\infty} \right). \end{aligned}$$

By the radial symmetry of  $W$ , i.e.  $W(x) = W(|x|)$ , we have

$$W_k^\infty(s, y) = \frac{\iota_k}{(\lambda_k^\infty)^{3/2}} W_{\ell_k \mathbf{e}_1 + \beta \mathbf{e}_2} \left( \frac{y - (\ell_k \mathbf{e}_1 + \beta \mathbf{e}_2)s - \mathbf{y}_k^\infty}{\lambda_k^\infty} \right).$$

Therefore, the Lorentz transform with parameter  $\beta \mathbf{e}_2$  of  $\tilde{v} = \tilde{u} - [\tilde{W}_1^\infty + \tilde{W}_2^\infty]$  is  $v = u - [W_1^\infty + W_2^\infty]$  and to finish the proof of Theorem 1 in case (A), we only have to prove that, for  $S_\delta$  large enough,

$$\sup_{s > S_\delta} \|(v, \partial_s v)(s)\|_{\dot{H}^1 \times L^2} \lesssim \delta. \quad (5.15)$$

By (5.6) and the decay properties of  $W$ , we know that for  $S_\delta$  large,

$$\sup_{s > S_\delta} \|(v, \partial_s v)(s)\|_{(\dot{H}^1 \times L^2)(\Gamma_{A_\delta}(s))} \lesssim \delta. \quad (5.16)$$

We now concentrate on an estimate for  $v(s)$  close to the soliton centers.

First, we claim that for any  $\delta > 0$ , for any  $B > 1$ , for  $S_\delta(\delta, B)$  large enough, and any  $s_0 > S_\delta$ ,

$$\iint_{|s-s_0|+|y-(\ell_k \mathbf{e}_1 + \beta \mathbf{e}_2)s| < B} (|\nabla v|^2 + |\partial_s v|^2) dy ds \lesssim \delta. \quad (5.17)$$

Indeed, by change of variables,

$$\begin{aligned} & \iint_{|s-s_0|+|y-(\ell_k \mathbf{e}_1 + \beta \mathbf{e}_2)s| < B} |\partial_s v|^2 dy ds \\ &= \frac{1}{1-\beta^2} \iint_{|s-s_0|+|y-(\ell_k \mathbf{e}_1 + \beta \mathbf{e}_2)s| < B} \left| (\tilde{v}_t - \beta \tilde{v}_{x_2}) \left( \frac{s - \beta y_2}{\sqrt{1-\beta^2}}, y_1, \frac{y_2 - \beta s}{\sqrt{1-\beta^2}}, \bar{y} \right) \right|^2 dy ds. \end{aligned}$$

Changing variables in the integral on the right-hand side as follows (note that the Jacobian of the change of variable is 1)

$$t = \frac{s - \beta y_2}{\sqrt{1-\beta^2}}, \quad x_1 = y_1, \quad x_2 = \frac{y_2 - \beta s}{\sqrt{1-\beta^2}}, \quad \bar{x} = \bar{y},$$

we obtain, for some  $C = C(\delta)$ ,

$$\begin{aligned} & \iint_{|s-s_0|+|y-(\ell_k \mathbf{e}_1 + \beta \mathbf{e}_2)s| < B} |\partial_s v|^2 dy ds \\ & \lesssim \iint_{|t-s_0\sqrt{1-\beta^2}|+|x-\tilde{\ell}_k \mathbf{e}_1 t| < CB} (|\tilde{v}_t|^2 + |\tilde{v}_{x_2}|^2) dx dt \\ & \lesssim B \sup_{t > s_0\sqrt{1-\beta^2}-CB} \int (|\tilde{v}_t|^2 + |\tilde{v}_{x_2}|^2)(t) dx \lesssim \delta, \end{aligned}$$

for  $S_\delta(\delta, B)$  large enough by (5.4). Proceeding similarly for  $|\nabla v|^2$ , we obtain (5.17).

It follows from (5.17) and (5.16) that for any  $s_0 > S_\delta$ , there exists  $s_1 \in [s_0, s_0 + 1]$ , such that

$$\|(v, \partial_s v)(s_1)\|_{\dot{H}^1 \times L^2}^2 \lesssim \delta. \quad (5.18)$$

Now, we use the equation of  $v$  to obtain an energy estimate for all large time. Note that  $v$  satisfies

$$v_{tt} - \Delta v + f(v + W_1^\infty + W_2^\infty) - f(W_1^\infty) - f(W_2^\infty) = 0. \quad (5.19)$$

Using the equation of  $v$ , the properties of  $W_k^\infty$  and standard small data Cauchy theory (by Strichartz estimates, see e.g. section 2 of [14]), taking  $\delta > 0$  small enough, and for  $S_\delta$  large enough, we obtain from (5.18),

$$\sup_{[s_1-1, s_1+1]} \int (|\nabla v|^2 + |\partial_s v|^2)(s, y) dy ds \lesssim \delta.$$

Thus, (5.15) is proved.

This completes the proof of Theorem 1 in case (A).

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